

LOGIC BLOG 2014

EDITOR: ANDRÉ NIES

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- Rapid announcement of new results in logic
- putting up results and their proofs for further research
- archiving results that fit nowhere else
- sandbox for later publication in a journal.

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Part 1. Randomness via algorithmic tests

1. CCR 2014: OPEN QUESTIONS

Written by Rupert Hölzl and André Nies in June 2014.

1.1. Relativising computable randomness. Let CR denote the class of computably random subsets of \mathbb{N} ; these are the sets on which no computable martingale succeeds. Let MLR denote the class of Martin-Löf random sets.

Question 1.1 (van Lambalgen’s theorem for CR). *Let X be computably random, and let Y be computably random relative to X . Is $X \oplus Y$ computably random?*

Question 1.2 (Joseph Miller). *Let X be computably random. Is there an incomputable oracle A such that X is computably random relative to A ?*

Let $\text{High}(\text{CR}, \text{MLR})$ denote the class of oracles A such that $\text{CR}^A \subseteq \text{MLR}$. Every PA complete set is in $\text{High}(\text{CR}, \text{MLR})$ by [17, Lemma 5.1] and [16].

Question 1.3 (Joseph Miller, [16], implicit in [17]). *Is every set in $\text{High}(\text{CR}, \text{MLR})$ PA-complete?*

We write $A \leq_{\text{CR}} B$ if $\text{CR}^B \subseteq \text{CR}^A$. Clearly \leq_{T} implies \leq_{CR} .

Question 1.4 (Nies). *Is \leq_{CR} equivalent to \leq_{T} ?*

Miyabe has provided an affirmative answer to an analogous question for Schnorr randomness.

1.2. Lowness for Martin-Löf randomness. We discuss two superclasses of the low for ML-random sets, and ask whether they are strictly larger.

The first can be seen as an analog in computability theory of the notion of null-additivity in set theory (see [3, Ch. 2]). Let $A \triangle B$ denote symmetric difference of sets A, B .

Definition 1.5 (Kihara). We say that $A \subseteq \mathbb{N}$ is *ML-additive* if $Z \in \text{MLR}$ implies $Z \triangle A \in \text{MLR}$ for each Z .

Clearly, if a set is low for ML-randomness then it is ML-additive.

Question 1.6. *Is every ML-additive oracle low for ML-randomness?*

We could replace the map $X \rightarrow X \triangle A$ by a general A -computable measure preserving bijection Φ on Cantor Space. We say A is Φ -preserving if $Z \in \text{MLR}$ implies $\Phi(Z) \in \text{MLR}$ for each Z . If A is low for ML-randomness, then it is Φ -preserving for each such Φ . It is open whether the converse holds.

Kjos-Hanssen’s intuition is that a sufficiently sparse A might be ML-additive. In this connection, whether, to the contrary, each ML-random Z is derandomized by arbitrarily fast growing functions: for each function g , is there f dominating g such that $Z - \text{range}(f)$ is not ML-random?

Kihara and Miyabe have recent results characterising additivity for Schnorr randomness. For instance, they show this coincides with lowness for uniform Schnorr randomness.

We say that A is low for a ML-random set R if R is ML-random relative to A . The second class strengthens the property of being low for R .

Definition 1.7 (Yu). Let $A \subseteq \mathbb{N}$ and let R be ML-random. We say that A is *absolutely low for R* if for all sets Y ,

$$R \in \text{MLR}^Y \rightarrow R \in \text{MLR}^{Y \oplus A}.$$

Every low for ML-random is low for Ω .

Fact 1.8 (Stephan and Yu). *Let A be K -trivial. Then A is absolutely low for Ω .*

Proof. Suppose Y is low for Ω . By the low for Ω basis theorem relative to Y , there is $C \geq_T Y$ such that C is low for Ω and also C is PA-complete relative to Y . By a recent result of Simpson and Stephan, this implies that $A \leq_T C$. Therefore $A \oplus Y$ is low for Ω . \square

We ask whether the converse holds.

Question 1.9 (Yu). *Is every set that is absolutely low for Ω K -trivial?*

Definition 1.10 (Merkle, Yu). We say an oracle A is super-absolutely low for Ω if for each oracle Y which is low for Ω ,

$$\forall n (K^Y(n) = K^{A \oplus Y}(n) + O(1)).$$

Note that every such set is absolutely low for Ω by definition. In fact it must be low for K , by letting $Y = \emptyset$. We ask the converse.

Question 1.11 (Merkle, Yu). *Is every low for K set super-absolutely low for Ω ?*

Note (Apr. 2015) The previous two questions have now been solved in the affirmative by Greenberg, Miller, Monin, and Turetsky in a paper called “Two more characterizations of the K -trivials”, to appear in NDJFL.

1.3. Descriptive string complexity. We say that $A \subseteq \mathbb{N}$ is *infinitely often K -trivial* if $\exists^\infty n K(X \upharpoonright n) \leq^+ K(n)$. For a reference see [2], who show for instance that every truth-table degree contains a set of this kind. Bauwens has recently shown that i.o. K -trivial and i.o. C -trivial are incomparable notions.

Question 1.12 (Barnpalias). *Suppose that $\forall r \exists n > r K(X \upharpoonright n) \leq^+ K(r, n)$. Is X infinitely often K -trivial?*

2. TURETSKY: PARTIAL MARTINGALES FOR COMPUTABLE MEASURES

By Daniel Turetsky, March 2014.

Recall that martingales are only required to be defined on cylinders of positive measure.

Definition 2.1. If μ is a probability measure on 2^ω , a *martingale with respect to μ* is a partial function $M : 2^{<\omega} \rightarrow [0, \infty)$ satisfying:

- For all $\sigma \in 2^{<\omega}$, $M(\sigma * 0)\mu(\sigma * 0) + M(\sigma * 1)\mu(\sigma * 1) = M(\sigma)\mu(\sigma)$.
- For all $\sigma \in 2^{<\omega}$, if $\mu([\sigma]) > 0$, then $M(\sigma)$ is defined.

Jason Rute [23] asked if we need to allow martingales to be partial like this for computable randomness. The answer is yes.

Theorem 2.2. *There is a computable probability measure μ and an $X \in 2^\omega$ which is not computably μ -random, yet no total computable μ -martingale succeeds on X .*

Proof. We build μ and a computable μ -martingale N . The real X will then be a sort of “true path” through N . Let $\langle M_i \rangle_{i \in \omega}$ be an enumeration of all potential μ -martingales. That is, these are partial computable maps $M_i : 2^{<\omega} \times \omega \rightarrow \mathbb{Q}_{\geq 0}$ such that for every $\sigma \in 2^{<\omega}$, the sequence $\langle M_i(\sigma, n) \rangle_{n \in \omega}$ is a (partial) Cauchy name for a non-negative real. We will abuse notation and refer to M_i as a partial map from $2^{<\omega}$ to $\mathbb{R}_{\geq 0}$.

During the construction, we will choose various positive rationals $\epsilon(\tau)$ for $\tau \in 2^{<\omega}$. We will define

$$M_\tau = \sum_{\tau(i)=1} \epsilon(\tau \upharpoonright_i) M_i.$$

Thus each M_τ is a partial computable function, and if $\tau_0 \subseteq \tau_1$, $M_{\tau_0} \leq M_{\tau_1}$. Note that M_\emptyset is identically 0, and in particular is total computable.

If $Y = \{i \in \omega : M_i \text{ is a total computable } \mu\text{-martingale}\}$, our intention is to ensure that $\liminf_{n \rightarrow \infty} M_\tau(X \upharpoonright_n) < \infty$ for all $\tau \prec Y$. By the savings trick, this will suffice to show that no total computable μ -martingale succeeds on X .

Strategy for τ :

Each $\tau \in 2^{<\omega}$ will have a *coding location* $\sigma_\tau \in 2^{<\omega}$, and will have a *preferred bit* $b_\tau \in \{0, 1\}$. We will threaten to make $\mu(\sigma_\tau * b_\tau) = 0$. The important point is that while we are threatening, we are under no obligation to define N along extensions of $\sigma_\tau * b_\tau$. Nor must we make much in the way of definitions for μ along proper extensions of $\sigma_\tau * b_\tau$. Of course, we must uniformly give a Cauchy name for $\mu(\sigma)$ for all σ , but we can do this above $\sigma_\tau * b_\tau$ without imposing any actual restraint.

We explain what is meant by the above. Our threatening to make $\mu(\sigma_\tau * b_\tau) = 0$ takes the form of declaring $\mu(\sigma_\tau * b_\tau) \in [0, 2^{-s}]$ at stage s . Note that this fulfills all of our obligations to make a Cauchy name. We can simultaneously declare $\mu(\rho) \in [0, 2^{-s}]$ for all $\rho \supset \sigma_\tau * b_\tau$, and this also meets our obligations to make Cauchy names. However, this second declaration carries no new information; it is entirely implied by the fact that $\mu(\sigma_\tau * b_\tau) \in [0, 2^{-s}]$. If at some stage s we choose to stop threatening and instead declare $\mu(\sigma_\tau * b_\tau) = 2^{-s}$, our previous declarations for the $\mu(\rho)$ impose no restraint; we are as free to define $\mu(\rho)$ as we would be if we had made no declarations for it at all. In particular, any $\rho \supset \sigma_\tau * b_\tau$ and any b are available as our next coding location once we decide to stop threatening at $\sigma_\tau * b_\tau$. So we will choose coding locations where M_τ has not gained much capital.

We now describe the strategy for τ , given σ_τ and b_τ . Our strategy will be based on several inductive assumptions:

- $\mu(\sigma_\tau)$ has already been defined.
- $N(\sigma_\tau * b_\tau)$ has already been defined.
- For all $i < |\tau| - 1$ with $\tau(i) = 1$, $\epsilon(\tau \upharpoonright_i)$ is already defined. Thus M_{τ^-} is already defined. If $\tau(|\tau| - 1) = 1$, M_τ is not yet defined because we lack $\epsilon(\tau^-)$, but this is the only missing ingredient, while if $\tau(|\tau| - 1) = 0$, M_τ is already defined.
- $M_{\tau^-}(\sigma_\tau * b_\tau) < 2$.

The strategy is as follows:

- (1) Begin threatening $\mu(\sigma_\tau * b_\tau) = 0$. That is, at stage s , for every $\rho \supseteq \sigma_\tau * b_\tau$, declare $\mu(\rho) \in [0, 2^{-s}]$. Also declare $\mu(\sigma_\tau * (1 - b_\tau)) \in [\mu(\sigma_\tau) - 2^{-s}, \mu(\sigma_\tau)]$. The extensions $\rho \supset \sigma_\tau * (1 - b_\tau)$ are irrelevant, so make declarations for $\mu(\rho)$ consistent with the uniform distribution above $\sigma_\tau * (1 - b_\tau)$. The strategy moves on to the next step, but this process continues until stated otherwise.
- (2) If $\tau(|\tau| - 1) = 0$, skip to Step 5.
- (3) Wait for $M_{|\tau|-1}$ to make a declaration that $M_{|\tau|-1}(\sigma_\tau * b_\tau) < a$ for some positive rational a .
- (4) Define $\epsilon(\tau^-) = \frac{2^{-2|\tau|}}{a}$.
- (5) Wait for M_τ to be defined on all extensions of $\sigma_\tau * b_\tau$ of length $|\sigma_\tau| + 1 + 2|\tau| + 2$.
- (6) Cease threatening $\mu(\sigma_\tau * b_\tau) = 0$, and instead declare $\mu(\sigma_\tau * b_\tau) = 2^{-s}$, while $\mu(\sigma_\tau * (1 - b_\tau)) = \mu(\sigma_\tau) - 2^{-s}$. Continue to extend μ above $\sigma_\tau * (1 - b_\tau)$ uniformly (these strings are still irrelevant).
- (7) If there are two strings σ_0 and σ_1 both extending $\sigma_\tau * b_\tau$ and of length $|\sigma_\tau| + 1 + 2|\tau| + 2$, with $M_\tau(\sigma_j) < M_\tau(\sigma_\tau * b_\tau) + 2^{-2|\tau|}$ for $j \in \{0, 1\}$ and $\sigma_0^- \neq \sigma_1^-$, then do the following:
 - (a) Declare $\sigma_{\tau*j} = \sigma_j^-$ and $b_{\tau*j} = \sigma_j(|\sigma_j| - 1)$.
 - (b) For all ρ extending $\sigma_\tau * b_\tau$ with $\rho \not\supseteq \sigma_j^-$ for either j , define $\mu(\rho)$ uniformly.
 - (c) Define $N(\sigma_j^-) = N(\sigma_j^- * 0) = N(\sigma_j^- * 1) = 2^{2|\tau|} N(\sigma_\tau * b_\tau)$ for both j , and define $N(\rho) = 0$ for all ρ extending $\sigma_\tau * b_\tau$ but incomparable with both σ_j^- . For all ρ extending $\sigma_j^- * (1 - b_{\tau*j})$, define $N(\rho) = N(\sigma_j^-)$.
 - (d) Begin the strategies for $\tau * 0$ and $\tau * 1$, and end the current strategy.
- (8) If there are no such σ_0 and σ_1 , define $\mu(\rho)$ and $N(\rho)$ uniformly on all extensions of $\sigma_\tau * b_\tau$ and end the current strategy.

Construction:

Begin by letting $\sigma_{\langle 0 \rangle} = \langle 0 \rangle$, $b_{\langle 0 \rangle} = 0$, $\sigma_{\langle 1 \rangle} = \langle 1 \rangle$ and $b_{\langle 1 \rangle} = 0$. Define $\mu(\langle \rangle) = 1$, $\mu(\langle 0 \rangle) = \mu(\langle 1 \rangle) = 1/2$. Define $N(\rho) = 1$ for $|\rho| < 3$, and $N(\rho) = 1$ for ρ with $\rho(1) = 1$. Begin the strategies for $\langle 0 \rangle$ and $\langle 1 \rangle$.

Verification:

Claim 2.3. *When we begin the strategy for τ , our inductive assumptions are met.*

Proof. This is easily seen to hold for the base case $|\tau| = 1$.

An examination of Step 7 shows that $\mu(\sigma_\tau)$ and $N(\sigma_\tau * b_\tau)$ will both be defined before the strategy for τ is begun. Inductively, by Step 4, we know that $\epsilon(\tau \upharpoonright_i)$ is defined for all $i < |\tau| - 1$ with $\tau(i) = 1$. It remains only to show that $M_{\tau^-}(\sigma_\tau * b_\tau) < 2$.

We argue by induction that $M_{\tau^-}(\sigma_\tau * b_\tau) < 2 - 2^{-|\tau^-|}$. Since $M_{\langle \rangle}$ is identically 0, clearly this holds for $|\tau| = 1$.

If $\tau(|\tau| - 1) = 0$, then

$$\begin{aligned} M_\tau(\sigma_\tau * b_\tau) &= M_{\tau^-}(\sigma_\tau * b_\tau) \\ &< 2 - 2^{-|\tau^-|} \\ &< 2 - 2^{-|\tau|} - 2^{-2|\tau|}, \end{aligned}$$

where the first inequality is by the inductive hypothesis.

If instead $\tau(|\tau| - 1) = 1$, then by our action at Step 4, we have that

$$\begin{aligned} M_\tau(\sigma_\tau * b_\tau) &= M_{\tau^-}(\sigma_\tau * b_\tau) + \epsilon(\tau^-)M_{|\tau|-1}(\sigma_\tau * b_\tau) \\ &< M_{\tau^-}(\sigma_\tau * b_\tau) + 2^{-2|\tau|} \\ &< 2 - 2^{-|\tau^-|} + 2^{-2|\tau|} \\ &\leq 2 - 2^{-|\tau|} - 2^{-2|\tau|}, \end{aligned}$$

where the second inequality is by the inductive hypothesis.

If we begin the strategies for $\tau * 0$ and $\tau * 1$, then by our action at Step 7, we know that $M_\tau(\sigma_{\tau*j} * b_{\tau*j}) < M_\tau(\sigma_\tau * b_\tau) + 2^{-2|\tau|}$, which completes the proof. \square

Claim 2.4. μ is a total computable measure.

Proof. By examination. If we threaten forever at $\mu(\sigma_\tau * b_\tau)$, then our action for τ ensures that μ is defined on all extensions of σ_τ . If we stop threatening, then our action for τ ensures that μ is defined on all ρ extending σ_τ with ρ not extending one of $\sigma_{\tau*0}$ or $\sigma_{\tau*1}$. We also begin the strategies for $\tau * 0$ and $\tau * 1$, who ensure that μ is defined on the extensions of $\sigma_{\tau*0}$ or $\sigma_{\tau*1}$, respectively. \square

Claim 2.5. N is a computable μ -martingale.

Proof. By examination. If we threaten forever at $\mu(\sigma_\tau * b_\tau)$, then we are not required to converge on extensions of $\sigma_\tau * b_\tau$. If we stop threatening and find σ_0 and σ_1 , then our action at Step 7 defines N for all ρ extending $\sigma_\tau * b_\tau$ except those extending σ_j , and then we begin the strategies for $\tau * j$, who will take responsibility for N above σ_j . Since we extended μ uniformly, the definition given precisely satisfies the martingale condition.

If we stop threatening but do not find σ_0 and σ_1 , then our action at Step 8 defines N for all ρ extending $\sigma_\tau * b_\tau$. Since the extension is uniform, the definition satisfies the martingale condition. \square

As before, let $Y = \{i \in \omega : M_i \text{ is a total computable } \mu\text{-martingale}\}$.

Claim 2.6. If $\tau \prec Y$, then we eventually begin the strategy for τ .

Proof. By induction on $|\tau|$. Since $\tau \prec Y$, M_τ is a total computable μ -martingale.

Clearly we eventually stop threatening at $\mu(\sigma_\tau * b_\tau)$. It suffices to show that the desired σ_0 and σ_1 exist. But suppose not. Then for all but at most 1 string σ of length $|\sigma_\tau| + 1 + 2|\tau| + 1$,

$$M_\tau(\sigma) \geq M_\tau(\sigma_\tau * b_\tau) + 2^{-2|\tau|}.$$

Since in this case we define μ uniformly by Step 8, we have

$$\begin{aligned}
\sum_{\sigma \in 2^{|\sigma_\tau|+1+2|\tau|+1}} \mu(\sigma) M_\tau(\sigma) &= \sum_{\sigma} 2^{-(2|\tau|+1)} \mu(\sigma_\tau * b_\tau) M_\tau(\sigma) \\
&\geq (2^{2|\tau|+1} - 1) \mu(\sigma_\tau * b_\tau) (M_\tau(\sigma_\tau * b_\tau) + 2^{-2|\tau|}) \\
&= \mu(\sigma_\tau * b_\tau) M_\tau(\sigma_\tau * b_\tau) (2^{2|\tau|+1} - 1) \left(1 + \frac{2^{-2|\tau|}}{M_\tau(\sigma_\tau * b_\tau)} \right).
\end{aligned}$$

Recalling that $M_\tau(\sigma_\tau * b_\tau) < 2$, consider

$$\begin{aligned}
(2^{2|\tau|+1} - 1)(1 + 2^{-2|\tau|-1}) &= 2^{2|\tau|+1} + 1 - 1 - 2^{-2|\tau|-1} \\
&= 2^{2|\tau|+1} - 2^{-2|\tau|-1} \\
&\geq 2^1 - 2^{-1} \\
&> 1.
\end{aligned}$$

But this contradicts the martingale condition. \square

Now, let $X = \bigcup_{\tau \prec Y} \sigma_\tau$. Since $\liminf M_\tau(X \upharpoonright_n) \leq 2$ for all $\tau \prec Y$, we see that no total computable μ -martingale succeeds on X . On the other hand, by construction we see that $N(\sigma_\tau) \geq 2^{2|\tau|-1}$ for all $\tau \prec Y$, and so N succeeds on Y . \square

3. KJOS-HANSEN: ANSWER TO A QUESTION OF BRENDLE ET AL. 2014

3.1. The first question, Question 4 (6). Here is a solution to Brendle, Brooke-Taylor, Ng and Nies [7], Question 4 (6). Kjos-Hanssen and Stephan worked on this on June 16, 2014 in Singapore. A is weakly meager engulfing if it computes a meager set that contains all recursive reals. This is equivalent to being (high or DNR).

Theorem 3.1. *There is weakly meager engulfing set that does not compute a Schnorr random and is of hyperimmune degree.*

We use a modification of Ted Slaman’s construction from the warm-up section of the paper “Comparing DNR and WWKL” [1]. There, one finds a construction of a function G which is DNR (hence weakly meager engulfing), recursive in $0'$ (hence hyperimmune), and computes no ML-random. To show it computes no Schnorr random it thus suffices to modify the construction so that G is not high. We do this by in fact making G low.

Ted Slaman mentioned (private communication in 2003 or 2004) that he thought the construction could be made low, but did not tell the proof. It turns out to be fairly easy:

Recall that G is a union of strings G_s . At stages $2e + 1$ we will take care of the requirements as in the original construction (making sure G computes no ML-random). At stages $s = 2e$ we work on the lowness requirements. We ask, is there a good or “bushy” tree starting from G_s consisting of strings making e in the jump of G ? If so, take an element of such a tree that’s also acceptable (which will always exist) and use it as G_{s+1} . If not, add to the list of acceptability requirements that there should not exist such a tree starting from any future G_t . This way we decide at say stage $2e$ whether e is

in the jump of G , hence in the end G is low. The acceptability also ensures that G is DNR.

4. HIGUCHI: INJECTIVE IMAGES OF COMPUTABLE RANDOMNESS

(By Kojiro Higuchi at Chiba University, Japan.) We give a short and self-contained proof of a result that can be seen as a special case of Rute [23, Thm. 7.9]. Let us use μ to denote the fair coin measure on the Cantor space $\{0, 1\}^\omega$. This paper gives a theorem such that the image of a computable random real under computable injection Φ over the Cantor space $\{0, 1\}^\omega$ is computable random with respect to the measure $\mu \circ \Phi^{-1}$.

Definition 4.1. Let $\Phi : \{0, 1\}^\omega \rightarrow \{0, 1\}^\omega$ be a computable function. A function $b : \{0, 1\}^{<\omega} \rightarrow \mathbb{R}_{\geq 0}$ is called Φ -martingale if b satisfies the equation

$$b(\tau)\mu(\Phi^{-1}(\tau\{0, 1\}^\omega)) = b(\tau 0)\mu(\Phi^{-1}(\tau 0\{0, 1\}^\omega)) + b(\tau 1)\mu(\Phi^{-1}(\tau 1\{0, 1\}^\omega))$$

for all $\tau \in \{0, 1\}^{<\omega}$. We define

$$\text{Succ}(b) = \{Y \in \{0, 1\}^\omega : \limsup_{n \rightarrow \infty} b(Y \upharpoonright n) = \infty\}.$$

We say that b *succeeds* on Y if $Y \in \text{Succ}(b)$.

Definition 4.2. Let $b : \{0, 1\}^{<\omega} \rightarrow \mathbb{R}_{\geq 0}$ be a Φ -martingale for a computable function $\Phi : \{0, 1\}^\omega \rightarrow \{0, 1\}^\omega$. We say that b satisfies the *saving property* if for all $k \in \omega$ and all $Y \in \text{Succ}(b)$, there exists $\tau \subsetneq Y$ such that $b(\rho) \geq k$ for any $\rho \in \tau\{0, 1\}^{<\omega}$.

We do not give a proof of the following lemma which tells us that we can always modify a given Φ -computable martingale to make it have the saving property without changing its succeeding set. A proof can be given just as a proof for the usual computable martingale.

Lemma 4.3. For any computable $\Phi : \{0, 1\}^\omega \rightarrow \{0, 1\}^\omega$ and any computable Φ -martingale $b : \{0, 1\}^{<\omega} \rightarrow \mathbb{R}_{\geq 0}$, there exists a computable Φ -martingale $b' : \{0, 1\}^{<\omega} \rightarrow \mathbb{R}_{\geq 0}$ with the saving property such that $\text{Succ}(b) = \text{Succ}(b')$. \square

Lemma 4.4. Let $\Phi : \{0, 1\}^\omega \rightarrow \{0, 1\}^\omega$ be a computable function and let $b : \{0, 1\}^{<\omega} \rightarrow \mathbb{R}_{\geq 0}$ be a Φ -martingale. Suppose that the limit

$$b'(\sigma) = \lim_{n \rightarrow \infty} \sum_{\tau \in \{0, 1\}^n} b(\tau) \frac{\mu(\sigma\{0, 1\}^\omega \cap \Phi^{-1}(\tau\{0, 1\}^\omega))}{\mu(\sigma\{0, 1\}^\omega)}$$

exists for all $\sigma \in \{0, 1\}^{<\omega}$. Then b' is a martingale. Furthermore,

$$\Phi^{-1}(\text{Succ}(b)) \subset \text{Succ}(b')$$

holds if b satisfies the saving property.

Proof. By assumption, b' is a function from $\{0, 1\}^{<\omega}$ into $\mathbb{R}_{\geq 0}$. Thus to show that b' is a martingale, we need to prove the equality for the martingales.

Fix $\sigma \in \{0, 1\}^{<\omega}$, We have the following equalities

$$\begin{aligned}
& b'(\sigma 0) + b'(\sigma 1) \\
&= \lim_{n \rightarrow \infty} \sum_{\tau \in \{0, 1\}^n} b(\tau) \frac{\mu(\sigma 0 \{0, 1\}^\omega \cap \Phi^{-1}(\tau \{0, 1\}^\omega)) + \mu(\sigma 1 \{0, 1\}^\omega \cap \Phi^{-1}(\tau \{0, 1\}^\omega))}{2^{-1} \mu(\sigma \{0, 1\}^\omega)} \\
&= 2 \lim_{n \rightarrow \infty} \sum_{\tau \in \{0, 1\}^n} b(\tau) \frac{\mu(\sigma \{0, 1\}^\omega \cap \Phi^{-1}(\tau \{0, 1\}^\omega))}{\mu(\sigma \{0, 1\}^\omega)} \\
&= 2b'(\sigma).
\end{aligned}$$

Thus, b' is a martingale.

Assume that b satisfies the saving property. Fix $X \in \{0, 1\}^\omega$ such that $\Phi(X) \in \text{Succ}(b)$ and fix $k \in \omega$. By the saving property of b , we can find a finite initial segment τ of $\Phi(X)$ such that $b(\rho) \geq k$ for all $\rho \in \tau \{0, 1\}^{<\omega}$. By the continuity of Φ , there exists a finite initial segment σ of X with $\Phi(\sigma \{0, 1\}^\omega) \subset \tau \{0, 1\}^\omega$. Now, we have the following inequalities

$$\begin{aligned}
b'(\sigma) &= \lim_{n \rightarrow \infty} \sum_{\rho \in \{0, 1\}^n} b(\rho) \frac{\mu(\sigma \{0, 1\}^\omega \cap \Phi^{-1}(\rho \{0, 1\}^\omega))}{\mu(\sigma \{0, 1\}^\omega)} \\
&= \lim_{n \rightarrow \infty} \sum \left\{ b(\rho) \frac{\mu(\sigma \{0, 1\}^\omega \cap \Phi^{-1}(\rho \{0, 1\}^\omega))}{\mu(\sigma \{0, 1\}^\omega)} : \rho \in \{0, 1\}^n, \rho \supset \tau \right\} \\
&\geq \lim_{n \rightarrow \infty} \sum \left\{ k \frac{\mu(\sigma \{0, 1\}^\omega \cap \Phi^{-1}(\rho \{0, 1\}^\omega))}{\mu(\sigma \{0, 1\}^\omega)} : \rho \in \{0, 1\}^n, \rho \supset \tau \right\} = k.
\end{aligned}$$

Since we fixed k arbitrarily, $\lim_{n \rightarrow \infty} b'(X \upharpoonright n) = \infty$. Thus $\Phi^{-1}(\text{Succ}(b)) \subset \text{Succ}(b')$ holds. \square

Lemma 4.5. *Let $\Phi : \{0, 1\}^\omega \rightarrow \{0, 1\}^\omega$ be a computable injection and let $b : \{0, 1\}^{<\omega} \rightarrow \mathbb{R}_{\geq 0}$ be a computable Φ -martingale. Then the function b' defined by*

$$b'(\sigma) = \lim_{n \rightarrow \infty} \sum_{\tau \in \{0, 1\}^n} b(\tau) \frac{\mu(\sigma \{0, 1\}^\omega \cap \Phi^{-1}(\tau \{0, 1\}^\omega))}{\mu(\sigma \{0, 1\}^\omega)}$$

is a (total) computable martingale.

Proof. Note that $\sigma \{0, 1\}^\omega$ and $\{0, 1\}^\omega \setminus \sigma \{0, 1\}^\omega$ are both compact. Thus their images under Φ are again compact (and therefore closed) by the continuity of Φ , and they are disjoint by the injectivity of Φ . Thus there exists a natural number $n \in \omega$ such that the sets $\{\Phi(X) \upharpoonright n : X \in \sigma \{0, 1\}^\omega\}$ and $\{\Phi(X) \upharpoonright n : X \in \{0, 1\}^\omega \setminus \sigma \{0, 1\}^\omega\}$ are disjoint. Note that such a natural number n can be found uniformly in a given $\sigma \in \{0, 1\}^{<\omega}$ since Φ is computable. If a natural n satisfies the property

$$(1) \quad \{\Phi(X) \upharpoonright n : X \supset \sigma\} \cap \{\Phi(X) \upharpoonright n : X \not\supset \sigma\} = \emptyset,$$

then so does $n + 1$. Moreover, if n satisfies (1), then the equalities

$$\begin{aligned}
 & \sum_{\tau \in \{0,1\}^n} b(\tau) \frac{\mu(\sigma\{0,1\}^\omega \cap \Phi^{-1}(\tau\{0,1\}^\omega))}{\mu(\sigma\{0,1\}^\omega)} \\
 &= \sum \left\{ b(\tau) \frac{\mu(\Phi^{-1}(\tau\{0,1\}^\omega))}{\mu(\sigma\{0,1\}^\omega)} : \exists X \supset \sigma, \tau = \Phi(X) \upharpoonright n \right\} \\
 &= \sum \left\{ b(\tau i) \frac{\mu(\Phi^{-1}(\tau i\{0,1\}^\omega))}{\mu(\sigma\{0,1\}^\omega)} : i \in \{0,1\}, \exists X \supset \sigma, \tau = \Phi(X) \upharpoonright n \right\} \\
 &= \sum_{\tau \in \{0,1\}^{n+1}} b(\tau) \frac{\mu(\sigma\{0,1\}^\omega \cap \Phi^{-1}(\tau\{0,1\}^\omega))}{\mu(\sigma\{0,1\}^\omega)}
 \end{aligned}$$

holds since b is a Φ -martingale. It follows that

$$b'(\sigma) = \sum_{\tau \in \{0,1\}^n} b(\tau) \frac{\mu(\sigma\{0,1\}^\omega \cap \Phi^{-1}(\tau\{0,1\}^\omega))}{\mu(\sigma\{0,1\}^\omega)}.$$

Since we can find a natural number n satisfying the above equation uniformly in a given σ as we noted, b' is a (total) computable function. Together with the previous lemma, we conclude that b' is a computable martingale. \square

Theorem 4.6. *Let $\Phi : \{0,1\}^\omega \rightarrow \{0,1\}^\omega$ be a computable injection. Then the images of computable random sequences under Φ are computable random w.r.t. the measure $\mu \circ \Phi^{-1}$.*

Proof. Fix $X \in \{0,1\}^\omega$ whose image under Φ is not computable random w.r.t. $\mu \circ \Phi^{-1}$. Choose a computable Φ -martingale b succeeding on $\Phi(X)$. By Lemma 4.3, we may safely assume that b satisfies the saving property. Define b' as

$$b'(\sigma) = \lim_{n \rightarrow \infty} \sum_{\tau \in \{0,1\}^n} b(\tau) \frac{\mu(\sigma\{0,1\}^\omega \cap \Phi^{-1}(\tau\{0,1\}^\omega))}{\mu(\sigma\{0,1\}^\omega)}$$

for all $\sigma \in \{0,1\}^{<\omega}$. By Lemma 4.4, we know that b' is a computable martingale. Finally, we have $\text{Succ}(b') \supset \Phi^{-1}(\text{Succ}(b))$ by Lemma 4.4. Thus b' succeeds on X since b does on $\Phi(X)$. It follows that X is not computable random. This completes the proof. \square

5. NIES: NORMALITY OF A REAL RELATIVE TO NON-INTEGRAL BASES, AND UNIFORM DISTRIBUTION

By André Nies, started March-April 2013. Represents work with Figueira.

5.1. Normality. For a real x let $\text{frac}(x)$ denote the fractional part $x - \lfloor x \rfloor$. (Some sources use $\{x\}$, which may stem from the time of typewriters when fonts and symbols were limited.) A sequence $\langle y_j \rangle_{j \in \mathbb{N}}$ of reals in $[0,1]$ is *uniformly distributed* if for each interval $[u,v] \subseteq [0,1]$, the proportion of $i < N$ with $y_i \in [u,v]$ tends to $v - u$ as $N \rightarrow \infty$. In other words, the Bernoulli($(v - u)$) distributed random variables $1_{[u,v]}(y_j)$ satisfy the law of large numbers.

Definition 5.1. Let $r > 1$. We say that $x \in \mathbb{R}$ is *normal in base r* if the sequence $\langle \text{frac}(xr^n) \rangle_{n \in \mathbb{N}}$ is uniformly distributed in $[0, 1]$.

For every $r > 1$, the set of reals that are normal for base r is conull. However, it is unknown whether, for instance, $1/2$ is normal in base $3/2$.

A real x is *absolutely normal* if it is normal in all integer bases > 1 . Let us say that x is *rationally normal* if it is normal in all rational bases > 1 .

In the following we will see that the rationally normal reals form a proper subset of the absolutely normal reals, even though they still are conull. This is a special case of a result by Brown, Moran and Pearce [8, Thm 2].

Sets $A, B \subseteq (1, \infty)$ are called *multiplicatively independent* (m.i.) if there are no $a \in A, b \in B, r, s \in \mathbb{N}$ such that $a^r = b^s$. For instance, $A = \mathbb{N} \setminus \{0, 1\}$ and $B = \{3/2\}$ are m.i. The result of Brown et al. says that given m.i. sets of algebraic numbers, every real is the sum of four numbers that are normal for all bases in A , but none in B . In particular, there are uncountably many reals that are absolutely normal, but not normal for the base $3/2$. Figueira and Nies [14] show that every polynomially random real is absolutely normal. This is also in upcoming work of Mayordomo and Hitchcock, who use polynomial dimension as an intermediate notion. The following likely conjecture would extend that result.

Conjecture 5.2. *Every polynomially random real is rationally normal.*

5.2. Uniform distribution and Weyl's theorem. Weyl (1916) proved that for every sequence $\langle a_i \rangle_{i \in \mathbb{N}}$ of distinct integers, for almost every $y \in [0, 1]$, the sequence $\langle \text{frac}(a_i y) \rangle_{i \in \mathbb{N}}$ is uniformly distributed. Avigad (Uniform distribution and algorithmic randomness, 2012) calls y *UD random* if this happens for every *computable* sequence of distinct integers. He shows that Schnorr randomness implies UD randomness; on the other hand, UD does not imply even Kurtz, so it is not an actual randomness notion.

Conjecture 5.3. *Suppose the function $1^j \rightarrow a_j$ is polynomial time computable. Then for every polynomially random real y , the sequence $\langle \text{frac}(a_i y) \rangle_{i \in \mathbb{N}}$ is uniformly distributed.*

Weyl's theorem can be extended to certain sequences $\langle u_i \rangle$ of distinct reals: if there is $k > 0$ such that $|u_i - u_j| > k$ for $i \neq j$, then $\langle \text{frac}(u_i y) \rangle_{i \in \mathbb{N}}$ is uniformly distributed for almost every y . See for instance [21, Cor 4.7] (first edition 1976), where it is derived from a more general theorem due to Koksma.

Joseph Miller, Figueira and Nies working at Miramar near Buenos Aires, have given an alternative proof, reducing the more general case to the case of integers. This proof is effective. In particular, for any computable sequence of reals as above, Schnorr randomness is sufficient. A feasible version of their proof should extend an affirmative answer to Conjecture 5.3 to the situation where a_j is a polynomial time computable real uniformly in 1^j .

We turn this Weyl's theorem extension into a notion of randomness:

Definition 5.4. x is *strongly UD-random* iff for every computable sequence (u_n) of rationals such that $(\exists k > 0)(\forall i, j) [i \neq j \rightarrow |u_i - u_j| > k]$, the sequence $(u_n x)$ is uniformly distributed modulo one.

Strong UD-random implies rational normality. Is strong UD-randomness a strictly stronger notion than UD-randomness?

Check connection with [15, 4], but observe that the transformation $T(x)$ defined as xq modulo one (for some fixed rational q) is not measure preserving, i.e. it is not true that $\lambda A = \lambda T^{-1}(A)$ for all measurable set A . For $q = 3/2$ and $A = [0, 1/2]$ we have $T^{-1}(A) = [0, 1/3] \cup [2/3, 1]$, and so $1/2 = \lambda A \neq \lambda T^{-1}(A) = 2/3$.

Let (x_n) be any sequence of reals. For $h \in \mathbb{Z}$ and $n \in \mathbb{N}$ define

$$S_h(n, x) = \frac{1}{n} \sum_{j < n} e^{2\pi i h x_j}.$$

Weyl's criterion states that the sequence (x_n) is uniformly distributed modulo one if and only if $\lim_{n \rightarrow \infty} S_h(n, x) = 0$ for all integers $h \neq 0$.

Observe that

$$\begin{aligned} |S_h(n, x)|^2 &= \frac{1}{n^2} \left(\sum_{j < n} e^{2\pi i h x_n} \right) \left(\sum_{j < n} e^{-2\pi i h x_n} \right) \\ &= \frac{1}{n^2} \left(n + \sum_{\substack{i, j < n \\ i \neq j}} e^{2\pi i h (x_i - x_j)} \right) \\ &= \frac{1}{n^2} \left(n + \sum_{i < j < n} e^{2\pi i h (x_i - x_j)} + e^{2\pi i h (x_j - x_i)} \right) \\ &= \frac{1}{n^2} \left(n + 2 \sum_{i < j < n} \cos(2\pi h (x_i - x_j)) \right), \end{aligned}$$

and so $|S_h(n, x)|^2$ is uniformly computable in h and n .

Recall that Schnorr randomness can be characterized in terms of total Solovay tests [10]. A *total Solovay test* is a sequence (A_n) of uniformly Σ_1^0 classes such that $\sum_n \lambda A_n$ is finite and a computable real. A set *passes* this test if it is in only finitely many of the A_n . A set is Schnorr random iff it passes all total Solovay tests.

We modify the proof of Avigad that Schnorr randomness implies UD-randomness to obtain the following:

Theorem 5.5. *Schnorr randomness implies strong UD-randomness.*

Proof. Suppose (u_n) is a sequence of rationals such that $(\forall i, j) [i \neq j \rightarrow |u_i - u_j| > k]$ for some $k > 0$ and that the sequence $(u_n x)$ is not uniformly distributed modulo one. Let $x_i = u_i x$. By the Weyl criterion, there is $h \in \mathbb{Z} \setminus \{0\}$ such that it is not true that $\lim_{n \rightarrow \infty} S_h(n, x) = 0$. Since for any m such that $n^2 \leq m < (n+1)^2$ we have $|S(n, x)| \leq |S(m^2, x)| + 2/\sqrt{n}$, we conclude that it is not true that $\lim_{n \rightarrow \infty} S_h(n^2, x) = 0$. Define

$$A_n = \{z \in [0, 1] \mid |S_h(n^2, z)|^2 > 1/\log n\}.$$

Suppose that $\delta > 0$ is such that there are infinitely many n such that $|S_h(n^2, x)|^2 > \delta$. For large enough n this implies that $|S_h(n^2, x)|^2 > 1/\log n$, and so $x \in A_n$ for infinitely many n , so it only remains to show that (A_n) is a total Solovay test.

By Markov's inequality, we have

$$\lambda A_n \leq \log n \int_0^1 |S_h(n^2, x)|^2 dx,$$

and by the proof of Koksma's General Metric Theorem in [21, Theorem 4.3],

$$\int_0^1 |S_h(n, x)|^2 dx \leq \frac{1}{n} + \frac{8 \log(3n)}{|h|kn}.$$

Hence

$$\sum_n \lambda A_n \leq \sum_n \log n \left(\frac{1}{n^2} + \frac{8 \log(3n^2)}{|h|kn^2} \right) < \infty.$$

It is clear that (A_n) is a sequence of uniformly Σ_1^0 classes, that λA_n is computable uniformly in n , and hence that $\sum_n \lambda A_n$ is computable. Therefore (A_n) is a total Solovay test. \square

Polynomial time randomness is incomparable to Schnorr randomness. On the one hand there are polynomial time random reals which are computable, and these cannot be Schnorr random. On the other hand, one can see from the proof of [11, Theorem 7.11.6] that there is a real which is Schnorr random and not polynomial time random.

6. RUTE: NO-RANDOMNESS-FROM-NOTHING FOR COMPUTABLE RANDOMNESS

(By Jason Rute, Pennsylvania State University.) Given a randomness notion, e.g. Schnorr randomness or Martin-Löf randomness, there are two properties that one would naturally expect to hold—randomness preservation and no-randomness-from-nothing. We will show the second holds for computable randomness.

Let μ be a computable probability measure on $\{0, 1\}^{\mathbb{N}}$. Assume $T: \{0, 1\}^{\mathbb{N}} \rightarrow \{0, 1\}^{\mathbb{N}}$ is a total computable map (later we will work with μ -a.e. computable maps), and that μ_T is the push-forward measure given by $\mu_T(A) = \mu(T^{-1}(A))$.

The first desirable property is *randomness preservation* or *randomness conversation*:

If $X \in \{0, 1\}^{\mathbb{N}}$ is μ -random, then $T(X)$ is μ_T -random.

The intuition behind this property is that if we use one random object X to generate another object $T(X)$ (for example, use a sequence of independent $1/2$ -weighted coin-flips into simulate a sequence of independent $1/4$ -weighted coin-flips), then $T(X)$ is random as well. This property is easy to prove for most of the standard randomness notions—such as Kurtz randomness, Schnorr randomness, Martin-Löf randomness, weak n -randomness, and n -randomness—as well as a few of the more exotic notions—such as difference randomness, Demuth randomness, and Oberwolfach randomness. However, it was shown by Bienvenu and Porter [5], as well as Rute [23] that it does not hold for computable randomness.

The second desirable property is *no-randomness-from-nothing*:

If $X \in \{0, 1\}^{\mathbb{N}}$ is μ_T -random, then $X = T(Y)$ for some μ -random $Y \in \{0, 1\}^{\mathbb{N}}$.

The idea behind this property is that all randoms in the push-forward measure come from a random source. Or conversely, one cannot condense a set of non-random objects into a random object. This property is also known as *surjectivity* since if randomness preservation holds, then no-randomness-from-nothing is equivalent to saying that T is a surjective map when the domain and range of T are restricted to the randoms. It is known that no-randomness-from-nothing holds for Martin-Löf randomness [5], weak- n randomness ($n \geq 2$) and n -randomness [the last two are personal communication from Laurent Bienvenu]. Bienvenu and Porter[5] asked if no-randomness-from-nothing also holds for computable randomness and Schnorr randomness.

Theorem (No-randomness-from-nothing for computable randomness.). If μ is a computable probability measure on $\{0, 1\}^{\mathbb{N}}$, $T: \{0, 1\}^{\mathbb{N}} \rightarrow \{0, 1\}^{\mathbb{N}}$ is a μ -a.e. computable map, and $Y \in \{0, 1\}^{\mathbb{N}}$ is μ_T -computably random, then $Y = T(X)$ for some μ -computably random $X \in \{0, 1\}^{\mathbb{N}}$.

This result can be generalized by replacing $\{0, 1\}^{\mathbb{N}}$ with other computable metric spaces and T with effectively measurable maps—Schnorr layerwise computable maps. These generalizations will appear in an upcoming paper. Also in that paper, I will give a very natural class of maps such that randomness preservation holds for computable randomness, and no-randomness-from-nothing holds for Schnorr randomness.

First some notation and definitions. Let ε be the empty string. For a measure μ on $\{0, 1\}^{\mathbb{N}}$ we will use the notation $\mu(\sigma) := \mu([\sigma])$ and for a measure μ on $\{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$, we will use the notation $\mu(\sigma \times \tau) := \mu([\sigma] \times [\tau])$. We say a measure μ on $\{0, 1\}^{\mathbb{N}}$ is computable if $\sigma \mapsto \mu(\sigma)$ is computable (and similarly for measures on $\{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$).

For computable randomness we will use an integral test characterization.

Definition 6.1 (Rute [23]). Let μ be a computable measure on $\{0, 1\}^{\mathbb{N}}$. A sequence $X \in \{0, 1\}^{\mathbb{N}}$ is μ -computably random iff $t(X) < \infty$ for all lower semicomputable functions $t: \{0, 1\}^{\mathbb{N}} \rightarrow [0, \infty]$ and all computable measures ν on $\{0, 1\}^{\mathbb{N}}$ such that

$$(2) \quad \int_{[\sigma]} t d\mu \leq \nu(\sigma) \quad (\sigma \in \{0, 1\}^*).$$

A similar definition can be used to define computable randomness on $\{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$.

Definition 6.2. Let μ be a computable measure on $\{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$. A pair $(X, Y) \in \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$ is μ -computably random iff $t(X, Y) < \infty$ for all lower semicomputable functions $t: \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}} \rightarrow [0, \infty]$ and all computable measures ν on $\{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$ such that

$$(3) \quad \int_{[\sigma] \times [\tau]} t d\mu \leq \nu(\sigma \times \tau) \quad (\sigma, \tau \in \{0, 1\}^*).$$

This next proposition justifies the previous definition.

Proposition 6.3. *Let μ be a computable measure on $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$. A pair $(X, Y) \in \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$ is μ -computably random iff $X \oplus Y$ is computably random on the pushforward measure of μ under the map $(A, B) \mapsto A \oplus B$.*

Proof. Call this pushforward measure μ' . Given any test pair t, ν satisfying (3) with μ , consider the test pair t', ν' where $t'(X \oplus Y) = t(X, Y)$ and ν' is the pushforward of ν under the map $(A, B) \mapsto A \oplus B$. Then t', ν' satisfies (2) with μ' . Conversely, any test pair t, ν satisfying (2) can be translated into a test pair t, ν satisfying (3) with μ . \square

Definition 6.4. A partial map $T: \{0,1\}^{\mathbb{N}} \rightarrow \{0,1\}^{\mathbb{N}}$ is said to be μ -a.e. computable for a probability measure μ if T is partial computable and $\mu(\text{dom } T) = 1$.

Lemma 6.5 (Rute [23]). *Let μ be a computable probability measure on $\{0,1\}^{\mathbb{N}}$, let $F: \{0,1\}^{\mathbb{N}} \rightarrow \{0,1\}^{\mathbb{N}}$ be a μ -a.e. computable map, and let $G: \{0,1\}^{\mathbb{N}} \rightarrow \{0,1\}^{\mathbb{N}}$ be a μ_F -a.e. computable map such that*

$$G(F(X)) = X \quad \mu\text{-a.e.} \quad \text{and} \quad F(G(Y)) = Y \quad \mu_F\text{-a.e.}$$

Then F and G both preserve computable randomness and if X is μ -computably random and Y is μ_F -computably random then

$$G(F(X)) = X \quad \text{and} \quad F(G(Y)) = Y.$$

Lemma 6.6. *Let μ be a computable probability measure on $\{0,1\}^{\mathbb{N}}$, let $T: \{0,1\}^{\mathbb{N}} \rightarrow \{0,1\}^{\mathbb{N}}$ be a μ -a.e. computable map, and let (id, T) be the map $X \mapsto (X, T(X))$. If (X, Y) is $\mu_{(\text{id}, T)}$ -computably random, then X is μ -computably random and $Y = T(X)$.*

Proof. Notice that (id, T) and its inverse $(X, Y) \mapsto X$ satisfy the conditions of Lemma 6.5. Therefore if (X, Y) is $\mu_{(\text{id}, T)}$ -computably random, then X is μ -computably random. And by the composition $(X, Y) \mapsto X \mapsto (X, T(X))$, we have that $Y = T(X)$. \square

The main idea of this blog entry is as follows. The measure $\mu_{(\text{id}, T)}$ is supported on the graph of T ,

$$\{(X, T(X)) : X \in \{0,1\}^{\mathbb{N}}\}.$$

Therefore, given a μ_T -computably random Y , by this last lemma, it is sufficient to find some X which makes (X, Y) $\mu_{(\text{id}, T)}$ -computably random. That is the goal of the rest of this article.

Lemma 6.7 (Folklore [11, Theorem 7.1.3]). *Assume $X \in \{0,1\}^{\mathbb{N}}$ is μ -computably random and ν is a computable measure. Then the following limit converges,*

$$\lim_n \frac{\nu(X \upharpoonright_n)}{\mu(X \upharpoonright_n)}.$$

Lemma 6.8. *Let μ be a computable probability measure on $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$ with second marginal μ_2 (that is $\mu_2(\tau) = \mu(\varepsilon \times \tau)$.) Assume $Y \in \{0,1\}^{\mathbb{N}}$ is μ_2 -computably random. The following properties hold.*

(1) For a fixed $\tau \in \{0, 1\}^*$, $\mu(\cdot \times \tau)$ is a probability measure,

$$\mu(\sigma 0 \times \tau) + \mu(\sigma 1 \times \tau) = \mu(\sigma \times \tau).$$

(2) For a fixed $\sigma \in \{0, 1\}^*$, $\mu(\sigma \times \cdot)$ is a probability measure.

(3) The following limit converges

$$\mu(\sigma|Y) := \lim_n \frac{\mu(\sigma \times Y \upharpoonright_n)}{\mu_2(Y \upharpoonright_n)}.$$

(4) The function $\mu(\cdot|Y)$ defines a probability measure

$$\mu(\sigma 0|Y) + \mu(\sigma 1|Y) = \mu(\sigma|Y).$$

(5) For a continuous function f , if $f^Y = f(\cdot, Y)$ then

$$\int f^Y d\mu(\cdot|Y) = \lim_n \frac{\int_{[\varepsilon] \times [Y \upharpoonright_n]} f d\mu}{\mu_2(Y \upharpoonright_n)}.$$

(6) For a nonnegative lower semicontinuous function t , if $t^Y = t(\cdot, Y)$ then

$$\int t^Y d\mu(\cdot|Y) \leq \lim_n \frac{\int_{[\varepsilon] \times [Y \upharpoonright_n]} t d\mu}{\mu_2(Y \upharpoonright_n)}.$$

Proof. (1) and (2) are apparent. Then (3) follows from (2) and Lemma 6.7. Then (4) follows from (1) and the definition of $\mu(\cdot|Y)$.

As for (5), first consider the case where f is a step function of the form $\sum_{i=0}^k a_i \mathbf{1}_{[\sigma_i]}$. This case follows from (4). Since such step functions are dense in the continuous functions under the norm $\|f\| = \sup_X f(X)$, we have the result.

As for (6), $t = \sum_k f_k$ for a sequence of continuous nonnegative f_k . Then we apply the monotone convergence theorem (MCT) for integrals and Fatou's lemma for sums,

$$\begin{aligned} \int t^Y d\mu(\cdot|Y) &= \int \sum_k f_k^Y d\mu(\cdot|Y) \stackrel{\text{MCT}}{=} \sum_k \int f_k^Y d\mu(\cdot|Y) \\ &= \sum_k \lim_n \frac{\int_{[\varepsilon] \times [Y \upharpoonright_n]} f_k d\mu}{\mu_2(Y \upharpoonright_n)} \stackrel{\text{Fatou}}{\leq} \lim_n \sum_k \frac{\int_{[\varepsilon] \times [Y \upharpoonright_n]} f_k d\mu}{\mu_2(Y \upharpoonright_n)} = \lim_n \frac{\int_{[\varepsilon] \times [Y \upharpoonright_n]} t d\mu}{\mu_2(Y \upharpoonright_n)}. \end{aligned}$$

□

Definition 6.9. Let μ be a measure on $\{0, 1\}^{\mathbb{N}}$ which may not be computable. A sequence $X \in \{0, 1\}^{\mathbb{N}}$ is *blind μ -Martin-Löf random* if $t(X) < \infty$ for all lower semicomputable t such that $\int t d\mu < \infty$.

Finally, we have the tools to find some X to pair with Y . What follows is a variation of van Lambalgen's theorem.

Lemma 6.10. Let μ be a computable measure on $\{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$. Let Y be μ_2 -computably random. Let X be blind $\mu(\cdot|Y)$ -Martin-Löf random. Then (X, Y) is μ -computably random.

Proof. Assume Y is μ_2 -computably random, but (X, Y) is not μ -computably random. Then there is a lower semicomputable function t and a computable measure ν such that $t(X, Y) = \infty$ and $\int_{[\sigma] \times [\tau]} t d\mu \leq \nu(\sigma \times \tau)$.

Let $t^Y = t(\cdot, Y)$. Then t^Y is lower semicomputable. Moreover, since t is lower semicontinuous, we have that

$$\int t^Y d\mu(\cdot|Y) \leq \lim_n \frac{\int_{[\varepsilon] \times [Y \upharpoonright_n]} t d\mu}{\mu_2(Y \upharpoonright_n)} \leq \lim_n \frac{\nu(\varepsilon \times Y \upharpoonright_n)}{\mu_1(Y \upharpoonright_n)}$$

where the right-hand side converges to a finite value by Lemma 6.7 since Y is μ_2 -computably random. Therefore, X is not blind Martin-Löf random since $\int t^Y d\mu(\cdot|Y) < \infty$ and $t^Y(X) = t(X, Y) = \infty$. \square

Theorem 6.11 (No-randomness-from-nothing for computable randomness.). *If μ is a computable probability measure on $\{0, 1\}^{\mathbb{N}}$, $T: \{0, 1\}^{\mathbb{N}} \rightarrow \{0, 1\}^{\mathbb{N}}$ is a μ -a.e. computable map, and $Y \in \{0, 1\}^{\mathbb{N}}$ is μ_T -computably random, then $Y = T(X)$ for some μ -computably random $X \in \{0, 1\}^{\mathbb{N}}$.*

Proof. Let $\nu = \mu_{(\text{id}, T)}$, and let X be blind $\nu(\cdot|Y)$ -Martin-Löf random (there are $\nu(\cdot|Y)$ -measure-one many, so there is at least one). By Lemma 6.10, (X, Y) is $\mu_{(\text{id}, T)}$ -computably random. By Lemma 6.6, X is computably random and $Y = T(X)$. \square

Part 2. Higher randomness

7. MONIN: SEPARATING HIGHER WEAKLY-2-RANDOM FROM Π_1^1 -RANDOM

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Theorem 7.1. *There is a higher weakly 2-random X that is not Π_1^1 -random.*

For background see the 2013 Logic Blog [13]

We say that a higher Δ_2^0 approximation $\{X_s\}_{s \in \omega_1^{ck}}$ of a sequence X is *wicked* if for any computable ordinal s we have that X is not in the closure of $\{X_t\}_{t < s}$. A sequence X is *wicked* if it has a *wicked* higher Δ_2^0 approximation. It is easy to prove that for every wicked sequence X we have $\omega_1^X > \omega_1^{ck}$:

Lemma 7.2. *Let $\{X_s\}_{s \in \omega_1^{ck}}$ be a wicked Δ_2^0 approximation of X . Then $\omega_1^X > \omega_1^{ck}$.*

Proof. Let $f: \omega \rightarrow \omega_1^{ck}$ be the $\Pi_1^1(X)$ function defined by setting $f(n)$ to the smallest ordinal s so that $X_s \upharpoonright_n = X \upharpoonright_n$. As the function is total it is $\Delta_1^1(X)$. The fact that the approximation is wicked directly implies that $\sup_n f(n) = \omega_1^{ck}$, and then $\omega_1^X > \omega_1^{ck}$. \square

The goal of this section is to show that there is a wicked weakly-2-random sequence. Let $\{S_i\}_{i \in \omega}$ be an enumeration of all the higher Σ_2^0 sets. For each S_i and each j let us define the Σ_1^1 closed set $F_{i,j}$ so that $S_i = \bigcup_j F_{i,j}$.

Sketch of the proof:

We will build the sequence X as a limit point of sequences $\{X_s\}_{s < \omega_1^{ck}}$. Each X_s is built as a limit of a sequence of strings $\{\sigma_s^n\}_{n < \omega}$ of bigger and bigger length.

At each stage we will ensure that X_s is in some sense *higher weakly-2-random at stage s* . What do we mean by this ? For some s and some n , as long as $\lambda(S_n[s]) = 1$, we believe that X_s should belong to $S_n[s]$. If at some point we have $\lambda(S_n[s]) < 1$ then n is removed from the set of indices that we use to make X_s weakly-2-random.

Concretely we have at each stage s a set of indices $\{e_n\}_{n \in \omega}$ which are initialized at stage 0 with $e_n = n$. Suppose that at stage s we have for each n that $\lambda(S_{e_n}[s]) = 1$. Then it is easy to build a Δ_1^1 sequence X_s in $\bigcap_n S_{e_n}[s]$:

We start by an arbitrary σ^0 and suppose that $\lambda(F_{e_0, d_0} \cap \sigma^0) \geq \varepsilon_0/2$. Assuming that for some n we have $\lambda(\bigcap_{k \leq n} F_{e_k, d_k} \cap \sigma^n) \geq \varepsilon_0/2$, we then continue recursively the construction as follow:

Step 1: We find one strict extention σ_s^{n+1} of σ_s^n so that $\lambda(\bigcap_{k \leq n} F_{e_k, d_k} \cap \sigma_s^{n+1})[s] \geq \varepsilon_{n+1}$.

Step 2: We search for a closed set $F_{e_{n+1}, d_{n+1}}$ inside $S_{e_{n+1}}$ so that $\lambda(\bigcap_{k \leq n+1} F_{e_k, d_k} \cap \sigma_s^{n+1})[s] \geq \varepsilon_{n+1}/2$.

This way we build the whole sequence X_s , and the measure requirement that we satisfy at each step allow us to conclude that $X_s \in \bigcap_n S_{e_n}[s]$. To have that the X_s converge to some X , we have to keep the chosen strings and closed sets at stage $s+1$ equal if possible to those of stage s . When do we have to change them ? three things can happend :

- (1) We might have $\lambda(S_{e_n})[s] = 1$ for all $s < t$ but $\lambda(S_{e_n})[t] < 1$.
- (2) We might have $\lambda(\bigcap_{k \leq n} F_{e_k, d_k} \cap \sigma^n)[t] \geq \varepsilon_n/2$ and $\lambda(\bigcap_{k \leq n} F_{e_k, d_k} \cap \sigma^{n+1})[s] \geq \varepsilon_{n+1}$ for all $s < t$ but $\lambda(\bigcap_{k \leq n} F_{e_k, d_k} \cap \sigma^{n+1})[t] < \varepsilon_{n+1}$
- (3) We might have $\lambda(\bigcap_{k \leq n} F_{e_k, d_k} \cap \sigma^{n+1})[t] \geq \varepsilon_{n+1}$ and $\lambda(\bigcap_{k \leq n+1} F_{e_k, d_k} \cap \sigma^{n+1})[s] \geq \varepsilon_{n+1}/2$ for all $s < t$ but $\lambda(\bigcap_{k \leq n+1} F_{e_k, d_k} \cap \sigma^{n+1})[t] < \varepsilon_{n+1}/2$

If (1) happends then the index e_n is setted to some index a so that $\lambda(S_a) = 1$ and then e_n can change at most once. If (2) happends it is the responsability of the string σ^{n+1} to change. If (3) happends it is the responsability of the index d_{n+1} to change.

In (2), it is easy to chose ε_{n+1} in a way that a string which fits the requirement always exists. Then it is enough to pick them in lexicographic order to be sure that we have one which fits the requirement after a finite number of change.

In (3) as long as $\lambda(S_{e_{n+1}}) = 1$ we are also sure that we will change only finitely often of index d_{n+1} . However if $\lambda(S_{e_{n+1}}) < 1$ it can happend that

d_{n+1} will change infinitely often at stage $s_1 < s_2 < \dots$ before we have at stage $t = \sup_n s_n$ that $\lambda(S_{e^{n+1}})[t] < 1$ (and then that e^{n+1} is set to a).

There is nothing we can do to prevent those infinitely many changes, which could lead as well to infinitely many changes of the string σ_{n+1} . However we can still ensure that if this happens, the string σ_n is banished, so that the final X is still wicked.

To do so, we need to chose the ε_n in a way that at least two extentions satisfying the requirements exists. So that if one is banished, still one remains. before starting the formal proof, we give a lemma which will help us to chose the right ε_n in the construction.

Lemma 7.3. *let σ be a string and F a closed set so that $\lambda(F \cap [\sigma]) \geq 2^{-|\sigma|-n}$. Then there is at least 2 extensions τ_1, τ_2 of σ of length $|\sigma| + n + 1$ so that for $i \in \{1, 2\}$ we have $\lambda(F \cap [\tau_i]) \geq 2^{-|\tau_i|-n-1}$.*

Proof. We have that $\lambda(F \cap [\sigma]) = \sum_{\tau_i} \lambda(F \cap [\tau_i])$. Suppose that for strictly less than 2 extensions of length $|\sigma| + n + 1$ we have $\lambda(F \cap [\tau_i]) \geq 2^{-|\tau_i|-n-1}$. Then we have:

$$\begin{aligned} \sum_{\tau_i} \lambda(F \cap [\tau_i]) &\leq 2^{-|\sigma|-n-1} + (2^{n+1} - 1)2^{-|\tau_i|-n-1} \\ &\leq 2^{-|\sigma|-n-1} + 2^{n+1}2^{-|\sigma|-2n-2} - 2^{-|\sigma|-2n-2} \\ &\leq 2^{-|\sigma|-n-1} + 2^{-|\sigma|-n-1} - 2^{-|\sigma|-2(n+1)} \\ &< 2^{-|\sigma|-n} \end{aligned}$$

which is a contradiction. \square

We now start the formal proof.

The construction:

Let $\{S_i\}_{i \in \omega}$ be an enumeration of all the higher Σ_2^0 sets. For each S_i and each j let us define the Σ_1^1 closed set $F_{i,j}$ so that $S_i = \bigcup_j F_{i,j}$.

Set $q_0 = 1$ and $m_0 = 1$ then recursively set $m_{n+1} = q_n + 2$ and $q_{n+1} = 2q_n + 4 - m_n$. Finally, let a be an integer so that $\lambda(S_a) = 1$.

At stage 0, at substage 0, set P_0^0 to be the set of strings of length m_0 , ordered lexicographically. Then set σ_0^0 to be the first string of P_0^0 , set $e_0^0 = 0$, set $d_0^0 = 0$. At substage $n + 1$, set P_0^{n+1} to be the set of strings of length m_{n+1} , ordered lexicographically. Then set σ_0^{n+1} to be the first string of P_0^{n+1} extending σ_0^n , set $e_0^{n+1} = n + 1$, set $d_0^{n+1} = 0$.

At successor stage $s+1$, set by convention $\sigma_{s+1}^{-1} = \epsilon$ and $F_{e_{s+1}^{-1}, d_{s+1}^{-1}} = 2^\omega$ (in order to avoid making a difference between substage 0 and other substages). At substage n (starting from 0), if $\lambda(S_{e_s^n})[s+1] = 1$ set $e_{s+1}^n = e_s^n$ and $P_{s+1}^n = P_s^n$, otherwise set e_{s+1}^n to be equal to a and $P_{s+1}^n = P_s^n - \{\sigma_s^n\}$. Then set σ_{s+1}^n to be the first string σ of P_{s+1}^n extending σ_{s+1}^{n-1} so that $\lambda(\bigcap_{k \leq n-1} F_{e_{s+1}^k, d_{s+1}^k} \cap [\sigma])[s+1] \geq 2^{-q_n}$.

Still at substage n , if $\lambda(\bigcap_{k \leq n} F_{e_s^k, d_s^k} \cap [\sigma_{s+1}^n])[s+1] \geq 2^{-q_n-1}$ set $d_{s+1}^n = d_s^n$. Otherwise set d_{s+1}^n to be the smallest d bigger than d_s^n so that $\lambda(\bigcap_{k \leq n-1} F_{e_{s+1}^k, d_{s+1}^k} \cap F_{e_{s+1}^n, d} \cap [\sigma_{s+1}^n])[s+1] \geq 2^{-q_n-1}$.

Finally after every substage, define A_{s+1} to be the unique element in $\bigcap_n [\sigma_{s+1}^n]$.

At limit stage s , compute a sequence of ordinals so that $s = \sup_m s_m$. By convention, like at successor stage set $\sigma_s^{-1} = \epsilon$ and $F_{e_{s+1}^{-1}, d_{s+1}^{-1}} = 2^\omega$. Then for each $n \geq 0$ set e_s^n to be the convergence value of $\{e_{s_m}^n\}_{m \in \omega}$ and set P_s^n to be the convergence value of $\{P_{s_m}^n\}_{m \in \omega}$. (among other things we will have to prove that we always have convergence).

At substage n , if $\{\sigma_{s_m}^n\}_{m \in \omega}$ does not converge set σ_s^n to be the first string of P_s^n extending $\sigma_{s_m}^{n-1}$, otherwise set σ_s^n to be the convergence value. If $\{d_{s_m}^n\}_{m < \omega}$ does not converge, set d_s^n to 0, otherwise set it to its convergence value.

Finally after every substage, define A_s to be the unique element in $\bigcap_n [\sigma_s^n]$.

The verification:

Claim 1: For every n the sequence $\{e_t^n\}_{t < \omega_1^{c_k}}$ can change at most once:

Fix some n and suppose that there is a stage s for which $e_{s+1}^n \neq e_s^n$. Let s be the smallest such stage. By construction we have that $e_{s+1}^n = a$ and as $\lambda(S_a) = 1$, we can prove by induction on stages that for any $t > s$ we have $e_t^n = a$. this implies that for the sequence $\{e_t^n\}_{t < \omega_1^{c_k}}$ can change at most once.

Claim 2: At every stage s the set $\{e_s^n \mid n \in \omega\}$ contains all the indices e such that $\lambda(S_e) = 1$.

It is true at stage 0 as $\{e_0^n \mid n \in \omega\} = \omega$. Suppose it is true at stage s . At stage $s+1$, if e_{s+1}^n is different from e_s^n it is by construction because $\lambda(S_{e_s^n})[s+1] < 1$. Then using induction hypothesis, Claim 2 is also true at stage $s+1$. At limit stage $s = \sup_n s_m$, we have using claim 1 that the sequence $\{e_{s_m}^n \mid n \in \omega\}_{s_m < s}$ converges to $\{e_s^n \mid n \in \omega\}_s$ and then by induction hypothesis we have that Claim 2 is true at limit stage s .

Claim 3: At every stage s the sequence set $\bigcap_n [\sigma_s^n]$ contains one and only one element A_s .

This is obvious by construction, as at any stage s and for any n the string σ_s^{n+1} extends σ_s^n .

Claim 4: For every stage s , any string τ of size m_n and any closed set F such that $\lambda(F \cap [\tau]) \geq 2^{-q_n-1}$, there is a string $\sigma \in P_{s+1}^{n+1}$ which extends τ

so that $\lambda(F \cap [\sigma]) \geq 2^{-q_{n+1}}$.

Suppose that $\lambda(F \cap [\tau]) \geq 2^{-q_n-1}$ for $|\tau| = m_n$. Then $\lambda(F \cap [\tau]) \geq 2^{-m_n-(q_n+1-m_n)}$. Using lemma 7.3 we have two strings τ_1 and τ_2 of length $m_n + (q_n + 1 - m_n) + 1 = q_n + 2 = m_{n+1}$ so that for $i \in \{1, 2\}$ we have $\lambda(F \cap [\tau_i]) \geq 2^{-(q_n+2)-(q_n+1-m_n)-1} = 2^{-2q_n-4+m_n} = 2^{-q_{n+1}}$. Also as τ_1 and τ_2 are of length m_{n+1} they belong to P_0^{n+1} . By construction and by Claim 1, at any stage s we have that P_0^{n+1} contains at most one more string than P_{s+1}^{n+1} . Then at any stage s we have at least one string $\sigma \in P_{s+1}^{n+1}$ which extends τ and so that $\lambda(F \cap [\sigma]) \geq 2^{-q_{n+1}}$.

Claim 5: The sequence A_s is *wicked*:

For a given n , let B_n be the set of stages s so that $\sigma_s^n = \sigma^n$ if the sequence $\{\sigma_s^n\}_{s < \omega_1^{ck}}$ converges to σ^n and $B_n = \emptyset$ if the sequence $\{\sigma_s^n\}_{s < \omega_1^{ck}}$ does not converge. Let us prove by induction on n that:

- (1) The set B_n is not empty.
- (2) The sequence of sets $\{d_t^k \mid k \leq n\}_{t \in B_n}$ can change at most finitely often.
- (3) The sequence of sets $\{\sigma_t^{n+1}\}_{t \in B_n}$ can change at most finitely often.

The sentence (1) is true for $n = -1$, as $\sigma_t^{-1} = \epsilon$ for every t . The sentence (2) is true for $n = -1$ as the the sequence $\{d_t^{-1}\}_{t < \omega_1^{ck}}$ never changes. The sentence (3) is true for $n = -1$ as by claim 1, the sequence $\{\sigma_t^0\}_{t < \omega_1^{ck}}$ can change at most once.

Let us suppose that (1), (2) and (3) are true up to n and let us show that (1) is true at level $n + 1$. By induction hypothesis (1) at level n we have a stage s so that for all $t > s$ we have $\sigma_t^n = \sigma_s^n$. Also by induction hypothesis (3) at level n we have a stage $r > s$ so that for all $t > r$ we have $\sigma_r^{n+1} = \sigma_t^{n+1}$. Then (1) is true at stage $n + 1$.

Let us suppose that (1), (2) and (3) are true at level n , that (1) is true at level $n + 1$ and let us show that (2) is true at level $n + 1$. By induction hypothesis (2) at level n and the fact that $B_{n+1} \subseteq B_n$ we have that :

- (4) The sequence of sets $\{d_t^k \mid k \leq n\}_{t \in B_{n+1}}$ can change at most finitely often.

By definition of B_{n+1} we have that:

- (5) The sequence $\{\sigma_t^{n+1}\}_{t \in B_{n+1}}$ can change at most finitely often.

Also by claim 1 we have that:

- (6) The sequence of sets $\{e_t^k \mid k \leq n+1\}_{t \in B_{n+1}}$ can change at most finitely often.

Suppose for contradiction that the sequence $\{d_t^{n+1}\}_{t \in B_{n+1}}$ changes infinitely often. From (4), (5) and (6) we can deduce that there are two stages $r_1 < r_2$ so that :

- (7) The sequence of sets $\{d_t^k \mid k \leq n\}_{t \in [r_1, r_2] \cap B_{n+1}}$ never changes.
- (8) The sequence of sets $\{e_t^k \mid k \leq n+1\}_{t \in [r_1, r_2] \cap B_{n+1}}$ never changes.
- (9) The sequence $\{\sigma_t^{n+1}\}_{t \in [r_1, r_2] \cap B_{n+1}}$ never changes.
- (10) The sequence $\{d_t^{n+1}\}_{t \in [r_1, r_2] \cap B_{n+1}}$ changes infinitely often.
- (11) We have $r_1 \in B_{n+1}$ and we have that r_2 is the smallest stage bigger than r_1 and infinitely many stages in B_{n+1} such that (10) is true.

Let us call the unique values for (7), (8) and (9) respectively $\{d^k \mid k \leq n\}$, $\{e^k \mid k \leq n+1\}$ and σ^{n+1} (note that σ^{n+1} is the convergence value for the sequence $\{\sigma_t^{n+1}\}_{t < \omega_1^{e^k}}$). From (7), (8) and (9) we deduce that:

$$(12) \quad \lambda(\bigcap_{k \leq n} F_{e^k, d^k} \cap [\sigma^{n+1}])[t+1] \geq 2^{-q_{n+1}} \text{ for } t \in [r_1, r_2] \cap B_{n+1}.$$

As otherwise, by construction, at least one of the three sequences (7), (8) or (9) would not be constant. Also if $\lambda(S_{e^{n+1}})[r_2] = 1$, we deduce from (12) that there is necessarily some d so that for any $t \in [r_1, r_2] \cap B_{n+1}$ we have $\lambda(\bigcap_{k \leq n} F_{e^k, d^k} \cap F_{e^{n+1}, d} \cap [\sigma^{n+1}])[t+1] \geq q_{n+1}/2$. But using (10) we know that no such d exists, as otherwise, by construction, the sequence of (10) would not change infinitely often. Then we have that $\lambda(S_{e^{n+1}})[r_2] < 1$. Also r_2 is the smallest such stage, as otherwise, by construction the sequence (8) would not be constant. Also from (9), by construction and the fact that r_2 is a limit stage (by definition of r_2) we have that $\sigma^{n+1} = \sigma_{r_2}^{n+1}$ and then that $r_2 \in B_{n+1}$. Then still by construction we have that $P_{r_2+1}^{n+1}$ does not contain $\sigma_{r_2}^{n+1}$ at stage r_2+1 , wich contradicts the fact that $r_2 \in B_{n+1}$. Then (2) is true at level $n+1$.

Suppose now that (1), (2) and (3) are true at level n , suppose also that (1) and (2) are true at level $n+1$ and let us show that (3) is true at level $n+1$. Suppose for contradiction that the sequence $\{\sigma_t^{n+2}\}_{t \in B_{n+1}}$ changes infinitely often. From induction hypothesis (2) at level $n+1$ we have that:

- (13) The sequence of sets $\{d_t^k \mid k \leq n+1\}_{t \in B_{n+1}}$ can change at most finitely often.

As before we have that (5) is true and that (6) is true, this time for $k \leq n+2$. We can deduce from that and from (13) that there are two stages

$r_1 < r_2$ so that :

(14) The sequence of sets $\{d_t^k \mid k \leq n+1\}_{t \in [r_1, r_2] \cap B_{n+1}}$ never changes.

(15) The sequence of sets $\{e_t^k \mid k \leq n+2\}_{t \in [r_1, r_2] \cap B_{n+1}}$ never changes.

(16) The sequence $\{\sigma_t^{n+1}\}_{t \in [r_1, r_2] \cap B_{n+1}}$ never changes.

(17) The sequence $\{\sigma_t^{n+2}\}_{t \in [r_1, r_2] \cap B_{n+1}}$ changes infinitely often.

Let us call the unique values for (14), (15) and (16) respectively $\{d^k \mid k \leq n+1\}$, $\{e^k \mid k \leq n+2\}$ and σ^{n+1} . From (14), (15) and (16) we deduce that:

(18) $\lambda(\bigcap_{k \leq n+1} F_{e^k, d^k} \cap [\sigma^{n+1}])[t+1] \geq 2^{-q_{n+1}-1}$ for $t \in [r_1, r_2] \cap B_{n+1}$.

As otherwise, by construction, at least one of the three sequences (14), (15) or (16) would not be constant. Now, using Claim 4 with $\bigcap_{k \leq n+1} F_{e^k, d^k}[r_2]$ as the closed set F , we deduce that there is at least one string $\sigma \in P_{r_2}^{n+2}$ so that for any $t \in [r_1, r_2] \cap B_{n+1}$ we have $\lambda(\bigcap_{k \leq n+1} F_{e^k, d^k} \cap [\sigma])[t+1] \geq 2^{-q_{n+2}}$ which directly contradicts (17).

We now use (1) and (3) in order to prove Claim 5. By (1) we have that X_s converges to X . By (3) we have for each n that the sequence $\{\sigma_t^{n+1} \mid \sigma_t^{n+1} \succ \sigma^n\}_{t < \omega_1^{c_k}}$ can change at most finitely often. Suppose now for contradiction that there is some t so that X is a limit point of $\{X_s\}_{s < t}$.

Then using Claim 6 (see further) we have that X is weakly-2-random and then not Δ_1^1 , so $X_t \neq X$. This implies that for some n we have infinitely many stages $s_0 < s_1 < s_2 < \dots$ so that $\sigma_{2i}^n = X \upharpoonright_{m_n}$ and $\sigma_{2i+1}^n \neq X \upharpoonright_{m_n}$. Let n be the smallest such integer. We have that $n \geq 1$ as $\{\sigma_0\}_{s < \omega_1^{c_k}}$ changes at most finitely often. Also by minimality of n we have some k so that $\sigma_{s_j}^{n-1} = X \upharpoonright_{m_{n-1}}$ for all $j \geq k$, which implies that the sequence $\{\sigma_t^n \mid \sigma_t^n \succ \sigma^{n-1}\}_{t < \omega_1^{c_k}}$ changes infinitely often. Thus X is wicked.

Claim 6: The sequence X is weakly-2-random:

At convergence we have a sequence of closed set $\{F^n = F_{e^n, d^n}\}_{n \in \omega}$ so that each F_{e^i, d^i} is a component of S_{e^i} . Consider also a sequence of ordinal $\{s_n\}_{n \in \omega}$, cofinal in $\omega_1^{c_k}$, and the closed set $A = \{X_{s_n}\}_{n < \omega} \cup \{X\}$. We have for any n and any stage t that the intersection of closed set $\bigcap_{s < t} \bigcap_{m \leq n} F_m[s] \cap A$ is not empty because by construction there is some stage s_k so that $X_{s_k} \in \bigcap_{m \leq n} F_m[t]$. Then also $\bigcap_{t < \omega_1^{c_k}} \bigcap_n F_n[t] \cap A$ is not empty, and as X is the only non Δ_1^1 point of A we have that X belongs to $\bigcap_n F_n$. This combined with Claim 2 allow us to conclude that X is weakly-2-random.

8. YU: A PROOF OF A RESULT BY KRIPKE ON THE UPWARD CLOSURE OF NULL SETS OF HYPER DEGREES

In [24], the following result is credited to Kripke.

Theorem 8.1. *Let A be a null set of hyperdegrees such that $\mathbf{0} \notin A$. Then the set $\mathcal{U}_h(A) = \{x \mid \exists y \in A (x \geq_h y)\}$ is also null.*

Theorem 8.1 significantly strengthened Sacks' result that the set of hyperdegrees hyperarithmetically above a fixed nonhyperarithmetic real is null. No published proof of the theorem was known. Here we give a short proof based on a recent result due to Chong and Yu, a hyperarithmetic version of Demuth's theorem on downward closure of random degrees.

Proof. In [9], it was proved that if x is Π_1^1 random and $y \leq_h x$, then $y \equiv_h y_0$ for some Π_1^1 random real y_0 . By partially relativizing the proof, we have that if x is $\Pi_1^1(z)$ random and $y \leq_h x$, then $y \equiv_h y_0$ for some $\Pi_1^1(z)$ random real y_0 .

Now suppose that A is a null set of hyperdegrees such that $\mathbf{0} \notin A$ and $\mathcal{U}_h(A) = \{x \mid \exists y \in A (x \geq_h y)\}$ is not null (may be even non-measurable!). Then there is some real z so that A is a subset of non- $\Pi_1^1(z)$ -random reals. Since $\mathcal{U}_h(A)$ is not null, it contains some real x which is $\Pi_1^1(z)$ -random. So there must be some $y \in A$ so that $y \leq_h x$. By Chong and Yu's result, there must be some $y_0 \equiv_h y$ so that y_0 is $\Pi_1^1(z)$ -random. But then $y_0 \in A$, a contradiction. \square

The higher Demuth's Theorem can also be used to construct a null maximal antichain of hyperdegrees and Turing degrees. This answers question 5.3 of Jockusch in [27]. Also it can be used to solve question 5.1 of Jockusch in [27].

9. YU: HIGHER RANDOMNESS OVER CONTINUOUS MEASURES

The following result is due to Chong and Yu.

Theorem 9.1. *$x \in \{z \mid z \in L_{\omega_1^x}\}$ if and only if x cannot be Π_1^1 -random relative to any continuous measure.*

By Theorem 9.1, if $x \notin L_{\omega_1^x}$, then x is Π_1^1 -random relative to some measure λ . However, the proof of Theorem 9.1 does not provide an explicit construction of λ provided $x \notin L_{\omega_1^x}$.

If x is hyperarithmetically equivalent to a Π_1^1 -random real, then we can find a constructive proof. By (and the proof of) higher Demuth Theorem and Reimann-Slaman Theorem, we have the following result.

Proposition 9.2. *For any real x , $x \equiv_h r$ for some Π_1^1 -random real r if and only if there is a hyperarithmetic continuous measure λ so that x is Π_1^1 -random relative to measure λ .*

The proof of Proposition 9.2 is constructive. In other words, if r is Π_1^1 random and $x \equiv_h r$, then λ can be constructed explicitly. So the proposition shed some light on the question whether there is a constructive proof of Theorem 9.1. Also it can be asked that whether there is a "level by level"

proof of Theorem 9.1. For example, if a continuous measure λ has a Π_1^1 -presentation, then which reals can be Π_1^1 -random relative to λ ?

In summary, we have the following question:

Question 9.3. *Is there a constructive proof of Theorem 9.1?*

The question can also be lifted to constructible level. We have the following result.

Proposition 9.4. *For any real x , $x \equiv_L r$ for some random real r over L if and only if there is a continuous measure λ in L so that x is random over L relative to measure λ .*

Then the further question is: Suppose that $ZF + AD + DC$, which reals can be $L[\hat{\lambda}]$ -random relative to a continuous measure λ ? Where $\hat{\lambda}$ is a presentation of λ .

Note that the set

$$A = \{x \mid x \text{ cannot be } L[\lambda]\text{-random for any continuous measure } \lambda\}$$

is Π_3^1 . So A is a Π_3^1 countable set under the assumption.

Proposition 9.5. $0^\# \in A$.

Proof. Assume that PD . By Simpson, there is a function $j : 2^\omega \rightarrow Ord$ (for example, $j(x) = (\kappa^+)^{L[x]}$) where $\kappa = \aleph_1$ so that

- (1) $x \leq_L y \rightarrow j(x) \leq j(y)$; and
- (2) $x \leq_L y \rightarrow (j(x) < j(y) \leftrightarrow x^\# \leq_L y)$.

So for any continuous measure $\lambda \not\geq_L 0^\#$, $\lambda \oplus 0^\# \geq_L \lambda^\#$. So $0^\#$ cannot be random over λ . □

So A is neither Π_2^1 nor Σ_2^1 .

By Proposition 9.4, A is closed under L -equivalence relation. However, A is not closed under Δ_3^1 -equivalence relation. For example, there is a Δ_3^1 L -random real Turing below $0^\#$.

Question 9.6. *Assuming $ZF + AD + DC$, give a characterization of A .*

10. KJOS-HANSEN AND YU: MIXED LOWNESS FOR HIGHER RANDOMNESS

All the results below were proved by Kjos-Hanssen and Yu in 2011 when Yu was visiting Kjos-Hanssen in San Diego.

The reals that are low for the implications from Martin-Löf randomness to Kurtz randomness and Schnorr randomness have in the past been characterized as the non-DNR degrees and the c.e. traceable degrees. Here we obtain the analogous results in the setting of hyperarithmetic reducibility: lowness for the implications from Π_1^1 -ML-randomness to Δ_1^1 -randomness and Δ_1^1 -Kurtz randomness equal Π_1^1 -traceable hyperdegrees and non- Π_1^1 -DNR hyperdegrees, respectively.

Definition 10.1. X is Δ_1^1 -SNR if there exists a function $f \leq_h X$ such that for each total Δ_1^1 function Φ , there do not exist infinitely many e with $f(e) = \Phi(e)$.

$\text{Low}(y, x)$	$\Delta_1^1\text{-Kurtz}$	$\Delta_1^1\text{-rand}$	$\Pi_1^1\text{-MLR}$
$\Delta_1^1\text{-Kurtz}$	$\Pi_1^1\text{-dominated, not } \Pi_1^1\text{-DNR [20]}$		
$\Delta_1^1\text{-rand}$	not $\Pi_1^1\text{-DNR}$	$\Pi_1^1\text{-traceable}$	
$\Pi_1^1\text{-MLR}$	not $\Pi_1^1\text{-DNR}$	$\Pi_1^1\text{-traceable}$	Δ_1^1

FIGURE 1. A complete picture of lowness for higher randomness between $\Pi_1^1\text{-ML}$ -randomness and $\Delta_1^1\text{-Kurtz}$ randomness.
Results from the present paper.

Definition 10.2. X is $\Pi_1^1\text{-DNR}$ if there exists a function $f \leq_h X$ such that there do not exist infinitely many e with $f(e) = \Phi_e(e)$. Here $\Phi_e(i) = R_e(i)$ is a universal Π_1^1 function. Let $R_e, e \in \omega$ be an effective list of all the partial recursive predicates. Then $R_e(i) = m \iff \forall X \exists n R_e(X, n, m, i)$.

Theorem 10.3. $\Delta_1^1\text{-SNR}$ implies $\Pi_1^1\text{-DNR}$.

Proof. Suppose X is not $\Pi_1^1\text{-DNR}$. So there are infinitely many e with $f(e) = \Phi_e(e)$.

Case 1: $X \geq_h O$. Impossible.

Case 2: $X \not\geq_h O$. Then $\omega_1^X = \omega_1^{\text{CK}}$. Then find e_1, e_2, \dots with $f(e_n) = \Phi_{e_n}(e_n) \downarrow$ by stage α . Bound $\alpha < \omega_1^{\text{CK}}$ (Σ_1^1 -bounding). Then $\exists^\infty e f(e) = \Phi(e) := \Phi_e(e)[\alpha]$. So X is not $\Delta_1^1\text{-SNR}$. \square

Theorem 10.4. Each $\Pi_1^1\text{-DNR}$ computes an infinite subset of a $\Pi_1^1\text{-MLR}$ set.

Proof. Follows the general plan of Barmpalias, Miller, Nies [6] (Miller's result). \square

Theorem 10.5. $\text{Low}(\Pi_1^1\text{-MLR}, \Delta_1^1\text{-random})$ equals $\Pi_1^1\text{-traceable}$.

Proof. This follows the method of Kjos-Hanssen, Nies, and Stephan 2005 [19]. \square

We also show $2 - \Pi_1^1\text{-DNR}$ implies $\Delta_1^1\text{-high}$ implies $\geq_h \Pi_1^1\text{-ML-random}$ implies $\geq_h \Pi_1^1\text{-DNR}$, and that the middle implication does not reverse.

We first prove that $2 - \Pi_1^1\text{-DNR}$ implies $\Delta_1^1\text{-high}$.

Lemma 10.6. If x is $2 - \Pi_1^1\text{-DNR}$ and $A \subseteq 2^\omega$ is a nonempty $\Sigma_1^1\text{-closed}$ set, then there must be some $z \leq_h x$ in A .

We may simulate the proof that every 2-DNR degree computes a member of every nonempty Π_1^0 -set to show Lemma 10.6.

Lemma 10.7. There is a nonempty $\Sigma_1^1\text{-closed}$ set A in which every member Turing computes all hyperarithmetic reals.

Proof. Note that there is a recursive function $f : \omega \rightarrow \omega$ so that for any $n \in \mathcal{O}$, $\Phi_{f(n)}$ is a tt -reduction so that $\Phi_{f(n)}^\mathcal{O} = H_n$.

Let $\sigma \in T$ if and only if $\sigma \in 2^{<\omega}$ and for any n , if $n \in \mathcal{O}$, then $\Phi_{f(n)}^\sigma \prec H_n$. So T is a Σ_1^1 -subtree of $2^{<\omega}$.

Note that $[T]$ is not empty since $\mathcal{O} \in [T]$.

If $x \in [T]$, then for any $n \in \mathcal{O}$, $\Phi_{f(n)}^x$ is total since $\Phi_{f(n)}$ is tt -reduction. Moreover, $\Phi_{f(n)}^x = H_n$.

Let $A = [T]$. □

Now if x is $2 - \Pi_1^1$, then by Lemma 10.6 and 10.7, there must be some real $z \leq_h x$ so that every hyperarithmetic real recursive in z . Then z must be Δ_1^1 -high.

To see that Δ_1^1 -high implies $\geq_h \Pi_1^1$ -ML-random, just note that every schnorr random relative to a Δ_1^1 -high real is Δ_1^1 -random. Then it follows by the dichotomy of randomness. No Π_1^1 -random can be Δ_1^1 -high. Otherwise, $r^{(\alpha)}$ Turing computes all hyperarithmetic reals for some Π_1^1 -random real r and recursive ordinal α . However, by the lowness of Π_1^1 -randomness, $r^{(\alpha)} \oplus \emptyset^{(\beta)} <_T r^{(\alpha+1)} \leq_T r \oplus \emptyset^{(\beta)}$ for some recursive ordinal β , a contradiction. So the reverse direction fails.

Clearly every Π_1^1 -ML-random hyperarithmetically computes a Π_1^1 -DNR function. ¹

¹Kihara proved that the reverse direction doesn't hold?

Part 3. Metric spaces

11. GAVRUSKIN AND NIES: LEFT-C.E. METRIC SPACES

Alexander Gavruskin and André Nies worked during a Tokerau Beach retreat Nov 2013. They finished their research a few months later. The paper has now appeared in the Lobchevskii Math. Journal.

Consider a Polish metric space $\mathcal{M} = (M, d, \langle p_k \rangle_{k \in \mathbb{N}})$ where d is the distance function, and $\langle p_k \rangle$ is a designated dense sequence. The p_i are called the special points. To say that \mathcal{M} is computable means that $d(p_i, p_k)$ is a computable real uniformly in i, k . There has been recent work on computable metric spaces (for instance by Melnikov), and on Π_1^0 equivalence relations on \mathbb{N} by Ianovski et al. [18]. The following generalizes both concepts.

Definition 11.1. We say that $\mathcal{M} = (M, d, \langle p_k \rangle_{k \in \mathbb{N}})$ is a left-c.e. [right-c.e.] metric space if $d(p_i, p_k)$ is a left-c.e. real [right-c.e. real] uniformly in numbers $i, k \in \mathbb{N}$. We write $d_s(p, q)$ for the distance of special points p, q at stage s .

We mainly study left-c.e. spaces. For them, intuitively speaking, the distance between points can increase over time.

Example 11.2. Let β be a left-c.e. real. Then $[0, \beta]$ is a left-c.e. metric space; so is the circle of radius β with the Euclidean metric inherited from \mathbb{R}^2 . In fact $[0, \beta]$ has a computable presentation even if β is noncomputable. (Note that this presentation is not effectively compact.) In contrast, the circle of radius β does not have a computable presentation at all if β is noncomputable. To see this, given n , find points q_0, \dots, q_{n-1} among the p_i such that $\forall i, k |d(q_i, q_{i+1}) - d(q_k, q_{k+1})| \leq 2^{-n}$ (where addition is taken mod n). This means the points q_i are the corners of an n -gon up to small error. Hence $\sum_{i < n} d(q_i, q_{i+1})$ converges to $2\pi\beta$ effectively. So $2\pi\beta$ is computable. (In more detail, each side of the n -gon has length within 2^{-n} of $2\beta \cos(\pi/n)$, so the error for the n -th approximation is at most $n2^{-n} + 2\beta|\pi - n \cos(\pi/n)|$.)

Let E be a Π_1^0 equivalence relation on \mathbb{N} . Define $d(x, y) = 1 - 1_E(\langle x, y \rangle)$, namely 0 if Exy and 1 otherwise. Clearly this metric makes \mathbb{N} a left-c.e. metric space (with $p_k = k$). This space has a computable presentation, though not uniformly. Note that the Π_1^0 equivalence relations can be uniformly identified with left-c.e. metric space where the set of possible distances is $\{0, 1\}$.

Definition 11.3. A *Cauchy name* in a metric space with distinguished dense sequence $\mathcal{M} = (M, d, \langle p_k \rangle_{k \in \mathbb{N}})$ is a function g on \mathbb{N} such that $d(p_{g(i)}, p_{g(k)}) \leq 2^{-i}$ for each $i \leq k$.

The Cauchy names in a left-c.e. metric space form a Π_1^0 subclass of Baire space. This together with the examples seems to suggest it is more natural to look at left-c.e. spaces, rather than at right-c.e. ones.

Definition 11.4. We say that an isometric embedding g from a left-c.e. [right-c.e.] metric spaces $\mathcal{M} = (M, d, \langle p_k \rangle_{k \in \mathbb{N}})$ to another $\mathcal{N} = (N, d, \langle q_k \rangle_{k \in \mathbb{N}})$ is *computable* if uniformly in k one can compute a Cauchy name for $g(p_k)$.

It is not hard to see that there is a universal object in the class of right-c.e. -metric spaces with respect to computable isometric embeddings. We

now obtain the corresponding result for left-c.e. spaces with a fixed bound on the diameter.

Theorem 11.5. *Let $\gamma > 0$ be a left-c.e. real. Within the class of left-c.e. metric spaces of diameter at most γ , there is a left-c.e. metric space \mathcal{U} which is universal with respect to computable isometric embeddings.*

Proof. For the duration of the proof, a left-c.e. pre-metric function will be a symmetric function $h: \mathbb{N} \times \mathbb{N} \rightarrow [0, \gamma]$ such that $h(v, v) = 0$ and $h(v, w)$ is a left-c.e. real, uniformly in v, w . Such a function is presented by its c.e. undergraph $G = \{\langle v, w, q \rangle : q \in \mathbb{Q}_0^+ \wedge v \leq w \wedge q \leq h(v, w)\}$. Let $h_t(x, y) = \max\{q : \langle v, w, q \rangle \in G_t\}$ so that $h(v, w) = \sup_t h_t(v, w)$.

Lemma 11.6. *Given a pre-metric function h , one can effectively determine a c.e. set W that is an initial segment of \mathbb{N} , and a pre-metric function g such that g satisfies the triangle inequality on W ; if h satisfies the triangle inequality then $W = \mathbb{N}$ and $g = h$.*

Proof of lemma. We extend the direct construction of a Π_1^0 -complete equivalence relation in [12, Subsection 13.1]. (Note that the proof in [18] differs from the original proof.)

Define a partial computable sequence of stages by $t_{-1} = -1$, $t_0 = 0$ and

$$(4) \quad t_{i+1} \simeq \mu t > t_i \quad \forall v, w \leq t_i [h_t(v, w) - \tilde{h}_t(v, w) \leq 2^{-i}] \wedge$$

$$(5) \quad \forall v, w \leq t_{i-1} [\tilde{h}_t(v, w) \geq \tilde{h}_{t_i}(v, w)],$$

where

$$\tilde{h}_t(v, w) = \inf \left\{ \sum_{r < t_i} h_t(q_r, q_{r+1}) : q_0, \dots, q_{t_i} \leq t_i \wedge q_0 = v \wedge q_{t_i} = w \right\}.$$

Note that $\tilde{h}_t(v, w) \leq h_t(v, w)$, and \tilde{h}_t satisfies the triangle inequality on $[0, t_i]$. Informally, the sequence of stages can only be continued at a stage $t > t_i$ if the “improved version” \tilde{h}_t that satisfies the triangle inequality is close enough to h_t ; furthermore, for values $v, w \leq t_{i-1}$ its value at t must be at least the one at the last relevant stage t_i .

If we define t_{i+1} we also put $(t_{i-1}, t_i]$ into W . Finally, for $v, w \in \mathbb{N}$, if $v \in W \wedge w \in W$ we let

$$g(v, w) = \sup \{ \tilde{h}_{t_{i+1}}(v, w) : v, w \leq t_i \},$$

and otherwise $g(v, w) = 0$.

Each $\tilde{h}_{t_{i+1}}$ satisfies the triangle inequality on $[0, t_i]$, so from the monotonicity in t_i it is clear that the pre-metric function g satisfies the triangle inequality on W .

Claim 11.7. *Suppose that h satisfies the triangle inequality. Then there are infinitely many stages t_i , so $W = \mathbb{N}$ and $g = h$.*

We proceed by induction on i . Suppose that t_i is defined. Assume that t_{i+1} remains undefined. Given $\epsilon > 0$ let $s > t_i$ be a stage such that $\forall x, y \leq t_i [h(x, y) - h_s(x, y) \leq \epsilon/t_i]$. Then for each $v, w \leq t_i$ we have $h(v, w) \leq \tilde{h}_s(v, w) + \epsilon$. Thus $\lim_{s > t_i} \tilde{h}_s(v, w) = h(v, w)$. This shows that both (4) and (5) are satisfied for sufficiently large s . Hence t_{i+1} will be defined,

contradiction. This concludes the inductive step and the proof of the lemma. \square

Clearly one can effectively list all the left-c.e. pre-metric functions as h_0, h_1, \dots . Let $\langle W_n, g_n \rangle_{n \in \mathbb{N}}$ be the pairs of a c.e. initial segment of \mathbb{N} and a pre-metric function and given by the lemma. Note that (W_n, g_n) presents a metric space for each n . The required universal left-c.e. metric space \mathcal{U} is an effective disjoint union of these spaces, where the distance between points in different components is γ . More formally, let f be a 1-1 computable function with range $\bigcup_n \{n\} \times W_n$. Let $p_k = k$ be the special points of \mathcal{U} , and define

$$\begin{aligned} d_{\mathcal{U}}(i, k) &= \gamma \text{ if } f(i)_0 \neq f(k)_0; \text{ otherwise, let} \\ d_{\mathcal{U}}(i, k) &= g_p(f(i)_1, f(k)_1), \text{ where } p = f(i)_0 = f(k)_0. \end{aligned}$$

Given a left-c.e. metric space $\mathcal{M} = (M, d, \langle q_r \rangle_{r \in \mathbb{N}})$, there is n such that $h_n(r, s) = d(q_r, q_s)$. This h_n satisfies the triangle inequality. So $q_r \rightarrow f^{-1}(\langle n, r \rangle)$ is a computable isometric embedding $\mathcal{M} \rightarrow \mathcal{U}$ as required. Note that the range of the embedding only contains special points of \mathcal{U} , which can be thought of as Cauchy names that are constant. \square

From the proof of the foregoing theorem for $\gamma = 1$, it is clear that $\{\langle i, k \rangle : d_{\mathcal{U}}(i, k) = 0\}$ is a universal Π_1^0 equivalence relation under computable reducibility in the sense of [18]. In this way we have re-obtained their existence result for such an object.

The bound γ on the diameter in Theorem 11.5 is necessary. If we discard that bound, there is no universal object in the class of all left-c.e. metric spaces with respect to computable isometric embeddings by the following fact.

Proposition 11.8. *Let $\mathcal{M} = (M, d, \langle p_k \rangle)$ be a left-c.e. metric space. There exists a left-c.e. ultrametric space \mathcal{A} such that \mathcal{A} can not be isometrically embedded into \mathcal{M} .*

Proof. We make \mathcal{A} a discrete metric space with domain consisting solely of special points q_0, q_1, \dots . We use the pair q_{2n}, q_{2n+1} to destroy the n -th potential computable embedding into \mathcal{M} . Such an embedding f would be given by computably associating to q_k a Cauchy name for $f(q_k)$. In particular, the map sending q_k to the first component of this Cauchy name is computable. Note that this component is a special point p_r of \mathcal{M} which has distance at most 1 from $f(q_k)$.

Let $\langle \phi_n \rangle_{n \in \mathbb{N}}$ be an effective listing of partial computable functions with domain an initial segment of \mathbb{N} . By the discussion above, it suffices to meet the requirements

$$R_n : \phi_n \text{ total} \Rightarrow d_{\mathcal{A}}(q_{2n}, q_{2n+1}) > d(\phi_n(q_{2n}), \phi_n(q_{2n+1})) + 2.$$

Construction. At stage s , for $n = 0, \dots, s$, if $\phi_n(q_{2n}), \phi_n(q_{2n+1})$ are defined, let

$$v_{n,s} = d_s(\phi_n(q_{2n}), \phi_n(q_{2n+1})) + 3;$$

otherwise $v_{n,s} = 0$. Set $d_{\mathcal{A},s}(q_{2n}, q_{2n+1}) = v_{n,s}$ towards satisfying R_n .

For each $n \neq k$, and $a = 2n, 2n+1, b = 2k, 2k+1$, set

$$d_{\mathcal{A},s}(q_a, q_b) = \max(v_{n,s}, v_{k,s}).$$

It is clear that the distances in \mathcal{A} remain finite. Also the ultrametric inequality holds because trivially for reals u, v, w we have

$$\max(u, w) \leq \max(\max(u, v), \max(v, w)).$$

We apply this to distances of the form $d_{\mathcal{A}}(q_{2n}, q_{2n+1})$ to get the ultrametric inequality between points from three different pairs. Finally, each requirement R_n is met. \square

The structure of our universal objects appears to be quite arbitrary. We ask whether an extra condition can be satisfied that would make them unique under computable isomorphism. This condition is effective homogeneity in the sense of Fraïssé theory.

Question 11.9. *Can a universal Π_1^0 equivalence relation, or a left-c.e. metric space, be effectively homogeneous?*

The bound γ on the diameter in Theorem 11.5 is necessary. If we discard that bound, there is no universal object in the class of all left-c.e. metric spaces with respect to computable isometric embeddings by the following fact.

Proposition 11.10. *Let $\mathcal{M} = \langle M, d, \langle p_k \rangle \rangle$ be a left-c.e. metric space. There exist a left-c.e. ultrametric space \mathcal{A} such that \mathcal{A} can not be isometrically embedded into \mathcal{M} .*

Proof. We make \mathcal{A} a discrete metric space with domain consisting solely of special points q_0, q_1, \dots . We use the pair q_{2n}, q_{2n+1} to destroy the n -th potential computable embedding into \mathcal{M} . Such an embedding f would be given by computably associate to q_k a Cauchy name for $f(q_k)$. In particular, the map sending q_k to the first component of this Cauchy name is computable. Note that this component is a special point p_r of \mathcal{M} which has distance at most 1 from $f(q_k)$.

Let $\langle \phi_n \rangle_{n \in \mathbb{N}}$ be an effective listing of partial computable functions with domain an initial segment of \mathbb{N} . By the discussion above, it suffices to meet the requirements

$$R_n : \phi_n \text{ total} \Rightarrow d_{\mathcal{A}}(q_{2n}, q_{2n+1}) > d(\phi_n(q_{2n}), \phi_n(q_{2n+1})) + 2.$$

Construction. At stage s , for $n = 0, \dots, s$, if $\phi_n(q_{2n}), \phi_n(q_{2n+1})$ are defined, let

$$v_{n,s} = d_s(\phi_n(q_{2n}), \phi_n(q_{2n+1})) + 3;$$

otherwise $v_{n,s} = 0$. Set $d_{\mathcal{A},s}(q_{2n}, q_{2n+1}) = v_{n,s}$.

For each $n \neq k$, and $a = 2n, 2n+1$, $b = 2k, 2k+1$, set

$$d_{\mathcal{A},s}(q_a, q_b) = \max(v_{n,s}, v_{k,s}).$$

It is clear that the distances in \mathcal{A} remain finite. Also the triangle inequality holds because trivially for reals u, v, w we have $\max(u, w) \leq \max(\max(u, v), \max(v, w))$. We apply this to distances $d_{\mathcal{A}}(q_{2n}, q_{2n+1})$. Finally, each requirement R_n is met. \square

12. GAVRUSKIN, NG AND NIES: LIMIT POINTS OF C.E. SETS OF SPECIAL POINTS IN A CMS

Gavruskin, Ng and Nies worked in Auckland. They generalised a result of LeRoux and Ziegler.

Theorem 12.1. *Let (X, d) be a perfect computable metric space with special points $\{q_i\}_{i \in \omega}$ and $\mathcal{C} \subseteq X$ be $\Pi_1^0(\emptyset')$ with respect to (X, d) . Then there is a c.e. set of special points $D \subseteq \{q_i\}_{i \in \omega}$ such that \mathcal{C} equals the set of limit points of D in (X, d) .*

Proof. We assume that $\{q_i\}_{i \in \omega}$ is a computable enumeration of the set of special points and fix a computable enumeration of all basic open balls $\{B_k\}_{k \in \omega}$, for instance, take $B_k = B(q_i, 2^{-j})$ where $k = \langle i, j \rangle$.

We also fix a Σ_2^0 enumeration of all basic open balls in $X - \mathcal{C}$, that is, we fix a Σ_2^0 -set $I \subseteq \omega$ such that $X - \mathcal{C} = \bigcup_{k \in I} B_k$. We now define the desired set D by doing the following for each $i \in \omega$:

Wait for some stage $t > i$ such that there exists some special point q satisfying:

- $q \in B_i$, and
- For each $j \leq i$ such that $B_j = B(q', 2^{-r})$ has not left $X - \mathcal{C}$ between stages i and t , we require that $d(q, q') > 2^{-r} - 2^{-i}$.

If q is found enumerate it in D , until we have enumerated 2^i many special points in D . We say that B_i contributes q to D .

It is easy to check that these conditions are Σ_1^0 , and so we produce a c.e. set D of special points. Note that since the space is perfect, each B_i either contributes nothing to D or it contributes exactly 2^i many special points to D .

We now verify that the set of limit points of D is \mathcal{C} .

First suppose that $x \in \mathcal{C}$. Since the space is perfect fix a sequence $\{q_{h(i)}\}_{i \in \omega}$ of distinct special points such that for every i , $d(q_{h(i)}, x) < 2^{-i}$. It suffices to show that for each i , the ball $B(q_{h(i)}, 2^{-i+1})$ contributes at least one point to D .

We proceed by contradiction. Suppose the ball $B_l = B(q_{h(i)}, 2^{-i+1})$ with $l = \langle h(i), i+1 \rangle$ contributes nothing to D . Since $q_{h(m)} \in B_l$ for all $m > i$, it follows that for each $m > i$, we must have the failure of the second condition for some $j \leq l$ such that B_j is permanently in the complement $X - \mathcal{C}$. Since there are only finitely many such $j \leq l$, there is some j such that $B_j = B(q', 2^{-r})$ is permanently in $X - \mathcal{C}$ where there are infinitely many $m > i$ such that $d(q_{h(m)}, q') \leq 2^{-r} - 2^{-l}$. However for each m we have

$$\begin{aligned} d(x, q') &\leq d(x, q_{h(m)}) + d(q_{h(m)}, q') \\ &\leq 2^{-m} + (2^{-r} - 2^{-l}) \\ &= 2^{-r} - (2^{-l} - 2^{-m}). \end{aligned}$$

By choosing a large enough m we conclude that $x \in B_j \subseteq X - \mathcal{C}$, a contradiction.

Now suppose that $x \notin \mathcal{C}$. Fix a j such that $x \in B_j = B(q, \delta)$. Suppose that B_j is enumerated permanently in the complement $X - \mathcal{C}$ at stage $s > j$. Fix $\varepsilon > 0$ such that $d(x, q) < \delta - \varepsilon$, and assume that $2^{-s} < \varepsilon$. We claim

that after stage s , no special point $p \in B(q, \delta - \varepsilon)$ can be enumerated in D for the sake of any B_k , $k > s$. Since each B_0, B_1, \dots, B_s contributes finitely many points to D , x cannot be a limit point of D .

Suppose on the contrary that at some stage larger than $s > j$ we have some special point $p \in B(q, \delta - \varepsilon)$ enumerated in D for the sake of B_k , for some $k > s$. By construction we must have $d(q, p) > \delta - 2^{-k} > \delta - 2^{-s} > \delta - \varepsilon$, a contradiction. \square

Part 4. Complexity of equivalence relations

13. TURETSKY: COMPLETE EQUIVALENCE RELATIONS

We work in the framework of m -reductions between arithmetical equivalence relations. For background see e.g. [18] and the references there.

Proposition 13.1. *For every oracle X , there is an equivalence relation on ω which is d.c.e. in X and effectively complete for such relations.*

Proof. Given a binary relation E which is d.c.e. in X , it suffices to uniformly construct an equivalence relation F which is d.c.e. in X and such that $F = E$ if E is an equivalence relation. We do this by copying E , but slowly (similar to the construction of a complete Π_1^0 -equivalence relation in [12, Subsection 13.1]). We keep a value n_s , which indicates how much of E we are copying.

Begin by defining $F_0 = \{(a, a) : a \in \omega\}$ and $n_0 = 0$.

At stage $s > 0$, if $E_s \upharpoonright_{n_{s-1}+1}$ is an equivalence relation, define

$$F_s = E_s \upharpoonright_{n_{s-1}+1} \cup \{(a, a) : a \in \omega\}$$

and $n_s = n_{s-1} + 1$. Otherwise, define $F_s = F_{s-1}$ and $n_s = n_{s-1}$.

Observe that F is d.c.e. as described, since the only time we change F 's value on a pair is when we are copying E , and E is d.c.e.. Further, it is easy to check by induction that every F_s is an equivalence relation, and so F is an equivalence relation. Finally, if E is an equivalence relation, n_s will grow without bound, and so we will have $F = E$. \square

Note that this construction works relative to any oracle, while the construction of a complete Π_1^0 equivalence relation does not. Indeed, since there is no complete Π_2^0 equivalence relation, that other construction cannot be relativized to \emptyset' . The difference between the two constructions is the reduction from E to F : when E is an equivalence relation, the reduction from E to F in this construction is the identity, and so is computable without X ; when E is an equivalence relation, the reduction from E to F in the Π_1^0 construction is a fast growing function that depends on the presentation of E .

Note also that this construction can be generalized to n -c.e. sets, while the construction for Π_1^0 equivalence relations can be generalized to co- n -c.e. sets. Similarly, the proof that there is no complete Π_2^0 equivalence relation can be generalized to show that there is no complete co- n -c.e. equivalence relation in \emptyset' . The key difference is that with n -c.e. sets, elements begin outside the set, while for co- n -c.e. sets, they begin in.

Part 5. Shen and Bienvenu: K -trivial, K -low and MLR-low sequences- a tutorial

These notes came out of lectures by L. Bienvenu at Poncelet laboratory, CNRS, Moscow, notes by A. Shen, LIRMM, Montpellier, CNRS, UM2, on leave from IITP RAS, Moscow.

A remarkable achievement in algorithmic randomness and algorithmic information theory was the discovery of the notions of K -trivial, K -low and Martin-Löf-random-low sets: three different definitions turns out to be equivalent for very non-trivial reasons. This paper, based on the course taught by one of the authors (L.B.) in Poncelet laboratory (CNRS, Moscow) in 2014, provides an exposition of the proof of this equivalence and some related results.

We assume that the reader is familiar with basic notions of algorithmic information theory (see, e.g., [25] for introduction and [26] for more detailed exposition). More information about the subject and its history can be found in [22, 11].

14. K -TRIVIAL SETS: DEFINITION AND EXISTENCE

Consider an infinite bit sequence and complexities of its prefixes. If they are small, the sequence is computable or almost computable; if they are big, the sequence looks random. This idea goes back to 1960s and appears in the algorithmic information theory in different forms (Schnorr–Levin criterion of randomness in terms of complexities of prefixes, the notion of algorithmic Hausdorff dimension). The notion of K -triviality is on the low end of this spectrum: we consider sequences that have prefixes of minimal possible (prefix) complexity:

Definition 14.1. A bit sequence $a_0a_1a_2\dots$, is called *K -trivial* if it has minimal possible prefix complexity of its prefixes, i.e., if

$$K(a_0a_1\dots a_{n-1}) = K(n) + O(1).$$

- Note that n can be reconstructed from $a_0\dots a_{n-1}$, so $K(a_0\dots a_{n-1})$ cannot be smaller than $K(n) - O(1)$.
- Every computable sequence is K -trivial (since $a_0\dots a_{n-1}$ can be computed given n).
- Similar definition for plain complexity has no sense, since this would imply that sequence A is computable (it is enough to have $C(a_0a_1\dots a_{n-1}) \leq \log n + O(1)$ for computability, see, e.g., [26, problems 48 and 49]).

With prefix complexity we have a weaker property:

Theorem 14.2. *Every K -trivial sequence is $\mathbf{0}'$ -computable.*

Here $\mathbf{0}'$ is the oracle for the halting problem.

Proof. Assume that $K((a)_n) = K(n) + O(1)$. Recall that $(a)_n = a_0a_1\dots a_{n-1}$ is equivalent to $(n, (a)_n)$, and use the formula for the complexity of a pair:

$$K((a)_n) = K(n, (a)_n) = K(n) + K((a)_n | n, K(n))$$

(with $O(1)$ -precision); this means that

$$K((a)_n | n, K(n)) = O(1).$$

So $(a)_n$ belongs to a $\mathbf{0}'$ -computable (given n) list of n -bit strings that has size $O(1)$. Therefore, a is a path in a $\mathbf{0}'$ -computable tree of bounded width and is therefore $\mathbf{0}'$ -computable. (Indeed, assume that the tree has k infinite paths that diverge before some level N . At levels after N we can identify all the paths, since all other vertices have finite subtrees above them, and we may wait until only k candidates remain.) \square

The existence of (non-computable) K -trivial sets is not obvious, but not very difficult, even if we additionally require the set to be enumerable.

Theorem 14.3. *There exists an enumerable undecidable K -trivial set.*

The existence of noncomputable K -trivial sequences was shown by Solovay in 1970s (answering Chaitin's question).

Proof. We want $K((a)_n)$ to be (almost) equal to $K(n)$. In terms of a priori discrete probability, we want to match $\mathbf{m}(n)$ at the vertex $(a)_n$. In game terms: when opponent (who “lower semicomputes” \mathbf{m}) increases the weight of n (the sum of his total weights never exceeds 1), we should increase the weight of some vertex (=string) of length n , achieving the same (up to $O(1)$ -factor) weight. Moreover, all these vertices should lie on a tree path that corresponds to a enumerable set, and the sum of our weights should be bounded by some constant.

It would be trivial for computable a : just fix some computable a and then place at $(a)_n$ the same weight as the opponent uses for n . But we want a to be non-computable. It is convenient to ensure that a is a characteristic function of a simple set in Post's sense.

Recall that a simple set A is a set with infinite complement that has non-empty intersection with every W_n that is infinite. Here by W_n we denote n -th enumerable set in some natural numbering of all c.e. sets. As in the classical Post's construction, we want for every n to put in A some element of W_n greater than $2n$, and then forget about W_n . (The bound $2n$ is needed to guarantee that the set has infinite complement.) Post did it without reservations (as soon as some element greater than $2n$ is discovered in W_n , it is added to A), but now, when such an element is found, we have to pay something for it: when we add some number u to A , the corresponding path a , the characteristic sequence for A , changes; the weight used on $(a)_u$ is lost, and should be recreated at the new place. Moreover, all the weights put at the extensions of $(a)_u$ are lost, too (since now these vertices are not on the path, and the new weights should be placed along the new path). This lost amount can be called the *cost* of the action.

Now we can explain the construction. Initially our set A is empty, and the corresponding path a is all zeros. While observing the growth of the discrete a priori probability $\mathbf{m}(n)$, we replicate the corresponding values along the path a . We also enumerate all W_n in parallel, and add some element $u \in W_n$ to A , if three conditions are satisfied:

- $u > 2n$;
- W_n was not satisfied before;
- the cost of this action is small (say, less than 2^{-n} , so the total cost for all n is bounded).

Here the cost of the action is the total weight placed at the vertices in a starting from level u (this weight is lost and should be replicated along the new path).

In this way the total weight used by us is bounded: the lost weight is bounded by $\sum 2^{-n}$, and the replicated weight is bounded by $\sum \mathbf{m}(n)$.

If W_n is infinite, it contains arbitrarily large elements, and the cost of adding u is bounded by

$$\mathbf{m}(u) + \mathbf{m}(u+1) + \mathbf{m}(u+2) + \dots,$$

which is guaranteed to go below threshold for large u . So every infinite W_n will be served at some moment. \square

This proof can be represented in a game form. In such a simple case this looks like an overkill, but the same technique is useful in more complicated cases, so it is instructive to look at this version of the proof. The game field consists of the set of the natural numbers (*lengths*), the full binary tree, and sets W_1, W_2, \dots (of natural numbers). The opponent increases the weights assigned to lengths: each length has some weight that is initially zero and can be increased by the opponent at any moment by any rational non-negative number; the only restriction is that the total weight (of all lengths) should not exceed 1. Also the opponent may add new elements to any of the sets W_i (initially they are empty). We construct a path a in the binary tree that is a characteristic sequence of some set A (initially empty) by adding elements to A ; we also increase the weights of vertices of the binary tree (in the same way as the opponent does for lengths); our total weight should not exceed 2.

One should also specify when the players can make moves; it is not important (the rules of the game always allow each player to postpone moves), but let us agree that the players make their moves in turns and every move is finite (finitely many weights of lengths and vertices are increased by some rational numbers, and finitely many new elements are added to W_i and A). This is the game with full information, the moves of one player are visible to the other one.

The game is infinite, and the winner is determined in the limit (assuming that both player obey the weight restrictions). Namely, we win if

- for the limit path a our weight of $(a)_n$ is not less than the opponent's weight of n ;
- for each n , if W_n is infinite, then W_n has a common element with A .

The winning strategy is as described: we match the opponent's weight along the current path, and also we add some u to A (changing the path and matching the opponent's weights along the new path) if u belongs to W_n , is greater than $2n$ and the loss (our total weight along the current path above u) does not exceed 2^{-n} .

This is a computable winning strategy. Indeed, the limit weights of all lengths form a converging series, so if W_n is infinite, it has some element that is greater than $2n$ and for which the loss (bounded by the tail of this series) is less than 2^{-n} .

Imagine now that we use this computable winning strategy against the "blind" computable opponent that ignores our moves and just enumerates

from below the a priori probability (as lengths' weights) and the sets W_i (the list contains all enumerable sets). Then the game is computable, our limit A is an enumerable simple set, and our weights for the prefixes of a (and therefore $\mathbf{m}((a)_n)$, since the limit weights form a lower semicomputable semimeasure) match $\mathbf{m}(n)$ up to $O(1)$ -factor.

15. K -TRIVIAL AND K -LOW SEQUENCES

Now we know that non-computable K -trivial sequences do exist. Our next goal is to show that they are “almost computable”. Namely, they are K -low in the sense of the following definition.

Consider a bit sequence a ; one can relativize the definition of prefix complexity using a as a oracle (the decompressor may use the values of a_i in its computation). For every oracle this relativized complexity K^a does not exceed (up to $O(1)$ additive term) the non-relativized prefix complexity, since the decompressor may ignore the oracle. But it can be smaller or not, depending on the oracle.

Definition 15.1. A sequence a is K -low if $K^a(x) = K(x) + O(1)$.

In other words, K -low oracles are useless for compression (better to say, decompression) purposes.

Obviously, computable oracles are low; the question is whether there exist non-computable low oracles. One can note that “classical” undecidable sets, like the halting problem, are not low: with oracle $\mathbf{0}'$ the table of complexities of all n -bit strings has complexity $O(\log n)$, but its non-relativized complexity is $n - O(1)$. (One can also consider the relativized and non-relativized complexities of the prefixes of Chaitin’s Ω -numbers.) Note also that K -low oracles are K -trivial (since $K^a((a)_n) = K^a(n) + O(1)$: the sequence a is computable in the presence of oracle a).

It turns out that the reverse implication is true, and this is quite surprising. For example, one may note that *the notion of a K -low sequence is Turing-invariant* (i.e., depends only on the computational power of the sequence) but for K -triviality there are no reasons to expect this (since the definition deals with prefixes).

On the other hand, it is easy to see that *if a and b are two K -trivial sequences, then their join* (the sequence $a_0b_0a_1b_1a_2b_2\dots$) *is also K -trivial*. Indeed, as we have mentioned, the K -triviality of a sequence a means that $K((a)_n | n, K(n)) = O(1)$; if also $K((b)_n | n, K(n)) = O(1)$, then (bound for the complexity of a pair) $K((a)_n, (b)_n | n, K(n)) = O(1)$, so

$$K(a_0b_0a_1b_1\dots a_{n-1}b_{n-1} | n, K(n)) = O(1),$$

and therefore $K(a_0b_0a_1b_1\dots a_{n-1}b_{n-1}) = K(n) + O(1)$. It remains to note that $K(n) = K(2n)$ and that we can extend the equality to sequences of odd length, since adding one bit changes the complexity of the sequence and its length only by $O(1)$. The similar result for low sequences is not obvious: if each of the sequences a and b separately do not change the complexity function, why their join is also powerless in this regard? Not obvious at all.

The proof of the equivalence (every K -trivial sequence is K -low) requires a rather complicated combinatorial construction. It may be easier to start

with a weaker statement and show that every K -trivial sequence is weaker (as an oracle) than the halting problem. (It follows from the equivalence statement, since $\mathbf{0}'$ is not K -trivial, see above.) This argument is given in the next section. On the other hand, the full proof is not so complicated, so the reader may also skip the next section and to read the equivalence proof without this training.

16. K -TRIVIAL SEQUENCE CANNOT COMPUTE $\mathbf{0}'$

In this section we prove that a K -trivial sequence cannot compute $\mathbf{0}'$, in the following equivalent version:

Theorem 16.1. *No K -trivial sequence can compute all enumerable sets.*

(Note that together with the existence result proved above it gives a solution of the Post problem, the existence of non-complete enumerable undecidable sets.)

Proof. The proof consists of several steps.

Game reformulation. We use the game argument and consider the following game with full information. The opponent approximates some sequence A by changing the values of Boolean variables a_0, a_1, a_2, \dots (so $A(i)$ is the limit value of a_i). He also assigns increasing weights to strings; the total weight should not exceed 1. We assume that initially all weights are zeros (and that the initial values of a_i are also zeros—just to be specific).

We assign increasing weights to integers (lengths); the sum of our weights is also bounded by 1. Also we construct some set W by (irreversibly) adding elements to it.

The opponent wins the game if

- each variable a_i is changed only finitely many times (so some limit sequence A appears);
- the (opponent's) limit weight of $(A)_i$, the i -bit prefix of A , is greater than (ours) limit weight of i , up to some multiplicative constant;
- the set W is Turing-reducible to A .

It is enough to construct a computable winning strategy in this game. Indeed, assume that some K -trivial A computes $\mathbf{0}'$. We know that then A is limit computable, so the opponent can computably approximate it, and at the same time approximate from below the a priori probabilities for all strings (ignoring our moves, as usual). Our strategy will then behave computably, generating some lower semicomputable semimeasure on lengths, and some enumerable set W . Then, according to the definition of the game, either this semimeasure is not matched by $\mathbf{m}((A)_i)$, or W is not Turing-reducible to A . In the first case A is not K -trivial; in the second case A is not Turing-complete.

Reduction to a game with fixed machine and constant. How can we win computably this game? First we consider a simpler game when the opponent has to declare in advance the constant c that relates the two semimeasures, and the machine Γ that reduces W to A . Imagine that we can win this game: for each c and Γ we have a (uniformly) computable strategy that

wins in the c - Γ -game (defined in a natural way). Since the constant c in the definition of c - Γ -game is arbitrary, we may use c^2 instead of c and assume by scaling that we can force the opponent to spend more than 1 while using only $1/c$ total weight and allowing him to match our moves up to factor c .

Now we mix the strategies for different c and Γ into one strategy. Note that two strategies that simultaneously increase weights of some lengths, can only help each other, so we need only to ensure that the sum of the increases made by all strategies is bounded by 1. More care is needed for the other part: each of the strategies constructs its own W , so we should isolate them. For example, to mix two strategies, we split \mathbb{N} into two parts N_1 and N_2 , say, odd and even numbers, and let the first/second strategy construct a subset of N_1/N_2 . Of course, each strategy should then beat not the machine Γ itself, but its restriction on N_i (the composition of Γ and the embedding of N_i into \mathbb{N}). In a similar way we can mix countably many strategies (splitting \mathbb{N} into countably many countable sets in a computable way).

It remains to consider some computable sequence $c_i \rightarrow \infty$ such that $\sum 1/c_i \leq 1$ and a computable sequence Γ_i that includes every machine Γ infinitely many times. (The latter is needed because we want every Γ to be beaten with arbitrarily large constant c .) Combining the strategies for these games as described, we get a computable winning strategy for the full game.

It is convenient to scale the c - Γ -game and require the opponent to match our weights exactly (without any factor) but allow him to use total weight c (instead of 1). We will prove the existence of the winning strategy by induction (the strategy for some c will be used to construct a strategy for larger c). To start, let us first construct the strategy for $c < 2$.

Winning a game with $c < 2$: strong strings. This winning strategy deals with some fixed machine Γ and ensures $\Gamma^A \neq W$ at one fixed point (say, 0); the other points are not used. Informally, we wait until the opponent puts a lot of weight on strings that imply $0 \notin \Gamma^A$. If this never happens, we win in one way; if this happens, we then add 0 to W and win in another way.

Let us explain this more formally. We say that a string u is *strong* if it (as a prefix of A) forces $\Gamma^A(0)$ to be equal to 0 (i.e., Γ outputs 0 on input 0 using only oracle queries in u). Simulating the behavior of Γ for different oracles, we can enumerate all strong strings. During the game we look at the following quantity:

*the total weight that our opponent has put on all (known)
strong strings.*

This quantity may increase because the opponent puts more weight or because we discover new strong strings, but never decreases. We try to make the opponent to increase this quantity (see below about how it is done). As we shall see, if he refuses, he loses the game (and the element 0 remains outside W). If this quantity becomes close to 1, we change our mind and add 1 into W . After that all the weight put on strong strings is lost: they cannot be the prefixes of A such that $\Gamma^A = W$, if $1 \in W$. So we can make our total weight equal to 1 in an arbitrary way (adding weight somewhere

if the total weight was smaller than 1), and the opponent needs to use additional weight 1 along some final A that avoids all strong strings, therefore his weight becomes close to 2.

Winning a game with $c = 1$: gradual increase. So our goal is to force the opponent to increase the total weight of strong strings. Let us describe our strategy as a set of substrategies (processes) that run in parallel. For each strong vertex x there is a process P_x ; we start it when we discover that x is strong. This process tries to force the opponent to increase the weight of some vertex above x (this vertex is automatically strong); we want to make the lengths of these strings different, so for every string x we fix some number l_x that is greater than $|x|$, the length of x .

Process P_x is activated when the current path goes through vertex x and sleeps otherwise (for example, it may never become active). So—when running—it always sees that the current path goes through x . The process gradually increases the weight of length l_x : it adds some small δ_x to the weight of l_x and waits until the opponent matches² this weight along the current path (whatever this path is), then increases the weight again by δ_x , etc. The values of δ_x are fixed for each x in such a way that $\sum_x \delta_x$ is small. The process repeats this increase until it gets a termination signal from the supervisor (see below when this happens). Note that at any moment the current path can change in such a way that x is no more on it; then P_x is put to sleep until x becomes on the current path again (and this may never happen).

The supervisor sends the termination signal to all the processes when (and if) the total weight they have used comes close to 1. After that the strategy adds 0 to W , as explained above.

Let us show that it is indeed the winning strategy. Consider a game where it is used. By construction, we do not violate the weight restriction. If the opponent has no limit path, he loses. If there is a limit path, but it does not have strong vertices on it, he also loses ($\Gamma^A(0) \neq 0$ for the limit A while $0 \notin W$). If x is a strong vertex on the limit path, the process P_x is active starting from some moment, so it can use all the weight (unless other processes do it earlier); in both cases the termination signal is sent. It remains to note that at this moment (when the termination signal is sent) the total weight put on strong string is close to 1: indeed, all the weight put by us is matched by the opponent, except for one portion of size δ_x for each vertex x ; the used weight is close to 1, and only small amount ($\sum_x \delta_x$) can be lost.

This argument shows how we can win the c - Γ -game for $c < 2$.

Induction statement. The idea of the induction step is simple: instead of forcing the weight increase for (some extension of) a strong vertex u directly, we can recursively call the described strategy at the subtree rooted at u (adding or not adding some other element to W instead of 0), and save our

²A technical remark: note that in our description of the game we have required that the opponent's weight along the path is strictly greater than our weight: if this is true in the limit, it happens on some finite stage.

weight (almost twice, since we know how to win the game with c close to 2). In this way we can win the game for arbitrary $c < 3$, and so on.

To be more formal, we consider a recursively defined process $P(k, x, \alpha, L, M)$ with the following parameters:

- $k > 0$ is a rational number, the required coefficient of weight increase;
- x is the root of the subtree where the process operates;
- $\alpha > 0$ is also a rational number, our “budget” (how much weight we are allowed to use);
- L is an infinite set of integers (lengths where our process may increase weight),³
- M is an infinite set of integers (numbers that our process may add to W).

The process can be started (or waked up) only when x is a prefix of the current path A , and is put to sleep when A changes and this is no more true, until x becomes the path of the current path again. It is guaranteed that P never violates the rules (about α , L , and M). Running in parallel with other processes (as a part of the game) and assuming that other processes do not touch lengths in L and numbers in M , the process $P(k, x, \alpha, L, M)$ guarantees (if not put into sleep forever or terminated externally) that one of the following possibilities happens:

- either limit path A does not exist,
- or $W \neq \Gamma^A$ for limit A ,
- or the opponent never matches some weight put on some length in L ,
- or the opponent spends more than $k\alpha$ weight on vertices above x with lengths in L .

Induction base. Now we can adapt the construction of the previous section and construct a process $P(k, x, \alpha, L, M)$ with these properties for arbitrary $k < 2$. For each y above x we select some $l_y \in L$ (different for different y), and select some positive δ_y such that $\sum \delta_y$ is small compared to the budget α . We choose some $m \in M$ and consider y (a vertex above x) as strong if it guarantees that $\Gamma^A(m) = 0$. Then for all strong y we start the process P_y that is activated when y is in the current path and increases the weight of l_y in δ_y -steps waiting until the opponent matches it. We terminate all the processes when (and if) the total weight used becomes close to α , and then add m to W (making all the weight placed by the opponent on strong vertices useless).

The restrictions are satisfied by the construction. Let us check the promised property. If there is no limit path, there is nothing to check. If the limit path does not go through x , we have no obligations (the process is put into sleep forever). Assume that the limit path goes through x and the process was not terminated externally. If there is no strong vertex on the limit path A , then $m \notin W$, but $\Gamma^A(m)$ is not 0, so $W \neq \Gamma^A$. If there is a strong vertex

³To use infinite sets as parameters, we should restrict ourselves to some class of infinite sets. For example, we may consider infinite decidable sets and represent them by programs enumerating their elements in the increasing order.

y on the limit path, then the process P_y was started and worked without interruptions (starting from some moment), so either some of the δ_y -increases was not matched (third possibility) or the termination signal was sent. In the latter case the total weight used was close to α , and after adding m to W it is lost, so either our weight is not matched or almost 2α is spent by the opponent above x (recall that all processes are active only when the current path goes through x).

Induction step. The induction step is similar: we construct the process $P(k, x, \alpha, L, M)$ in the same way as for the induction base. The difference is that instead of δ_y -increase in the weight of l_y the process P_y now recursively calls

$$P(k', y, \delta_y, L', M')$$

for vertex y with some smaller k' (one can take $k' = k - 0.5$), the budget δ_y and some $L' \subset L$ and $M' \subset M$. If the started process forces the opponent to spend more than $k'\delta_y$ on the vertices above y with lengths in L' , then a new process

$$P(k', y, \delta_y, L'', M'')$$

is started for some other $L'' \subset L$ and $M'' \subset M$, etc. All the subsets L', L'', \dots should be disjoint (and also disjoint for different y), as well as M', M'', \dots . So we should first of all split L into a sum of disjoint infinite subsets L_y parametrized by y and then split each L_y into $L'_y + L''_y + \dots$ (for the first, second, etc. recursive calls). The same is done for M , but here we in addition to the sets M_y select some m outside all M_y . This m is used to define strong vertices as those that guarantee $\Gamma^A(m) = 0$.

We start processes P_y (as described above: each of them makes a potentially infinite sequence of recursive calls with the same $k' = k - 0.5$ and budget δ_y) for all discovered strong vertices y (putting them into sleep when y is not on the current path). We take note of the total weight used by all P_y (for all y) and sent a termination signal to all P_y when this weight comes close to the threshold α (so it never crosses this threshold).

Let us show that we achieve the declared goal (assuming that the recursive calls achieve it). First, the restrictions about L , M and α are guaranteed by the construction. If there is no limit path, we have no other obligations. If the limit path exists but does not go through x , our process will be externally put to sleep, and again we have no obligations. So we may assume that the limit path goes through x and that our process is not terminated externally. If the limit path does not go through any strong vertex (defined using m), then $W \neq \Gamma^A$ for the limit path A , since $m \notin W$ and $\Gamma^A(m)$ does not output 0. If the limit path goes through some strong y , the process P_y will be active starting from some point, and makes recursive calls $P(k', y, \delta_y, L', M'), \dots, P(k', y, \delta_y, L'', M'')$, etc. Now we use the inductive assumption and assume that these calls achieve their declared goals. Consider the first call. If it succeeds by achieving one of three first alternatives, then we are done. If it succeeds by achieving the fourth alternative, i.e., by spending more than $k'\delta_y$ on the weights from L' , then the second call is made, and again either we are done or the opponent spends more than $k'\delta_y$ on the weights from L'' . And so on: at some point we either succeed globally, or exhaust the budget

and our main process sends the termination signal to all P_y . Then all the weight spent, except for the δ_y 's for the last call at each vertex, is matched by the opponent with factor k' , and on the final path the opponent has to match it with factor 1, so we are done (assuming that $k < k' + 1$ and $\sum \delta_x$ is small enough).

This finishes the induction step, so we can win every c - Γ -game by calling the recursive process at the root. We have proven that K -trivial sets do not compute $\mathbf{0}'$. \square

17. K -TRIVIAL SEQUENCES ARE K -LOW

Now we want to prove the promised stronger result:

Theorem 17.1. *All K -trivial sequences are K -low.*

Proof. Let us start with some preparations.

First, we already know that K -trivial sequences are $\mathbf{0}'$ -computable. So it is enough to show that every K -trivial $\mathbf{0}'$ -computable sequence is K -low.

Second, let us provide a convenient way to represent the a priori probability $\mathbf{m}^A(\cdot)$ with oracle A . (We need to show that it is not significantly larger than $\mathbf{m}(\cdot)$.) We may describe \mathbf{m}^A in the following way. The sequence A is a path in a full binary tree. Imagine that at every vertex of a tree there is a label of the form (i, η) where i is an integer, and η is a non-negative rational number. This label is read as “please add η to the weight of i ”. We assume that the labeling is computable. We also require that for every path in the tree the sum of all rational numbers along the path does not exceed 1. Having such a labeling, and a path A , we can obey all the labels along the path, and get a semimeasure on integers. This semimeasure is semicomputable with oracle A .

In fact, this construction is quite general: for every machine M with oracle generating a lower semicomputable discrete semimeasure, we can find a labeling that gives the same semimeasure (in the way described) for every oracle. Indeed, we may simulate the behavior of M for different oracles A , and look which part of A was read when some increase in the output semimeasure happens. This can be used to create a label at some tree vertex. We need to make the labeling computable; also, according to our assumption, each vertex adds weight only to one object, but both requirements can be easily fulfilled by postponing the weight increase (we push the queue of postponed requests up the tree). If the sum of the increase requests along some path becomes greater than 1, this means that for A extending this path we do not get a semimeasure. As usual, we can trim the requests and guarantee that we get semimeasures along all paths, not changing the existing semimeasures.

We may assume now that some computable labeling is fixed such that for every path A the resulting semimeasure (obtained by fulfilling all the requests along the path) is \mathbf{m}^A .

Game description. We prove this theorem by providing a winning strategy in some game. (If you have read the previous section, note that the part related to K -triviality is the same.) The opponent approximates some sequence A by changing the values of Boolean variables a_0, a_1, a_2, \dots and assigns increasing

weights to strings; the total weight should not exceed 1. We assume that initially all weights are zeros (and that the initial values of a_i are also zeros).

We assign increasing weights to integers (*lengths*); the sum of our weights is also bounded by 1. This semimeasure will be compared to the opponent's weights along his path. We also assign increasing weights to another type of integers, called *objects*: on them we compare our semimeasure with the semimeasure \mathbf{m}^A (determined by the opponent's limit path A)⁴.

The opponent wins the game if the following three conditions are satisfied:

- the limit sequence A exists;
- opponent's semimeasure along the path $*$ -exceeds our semimeasure on lengths, i.e., there exists some $c > 0$ such that for all i the opponent's weight of $(A)_i$ is greater than (our weight of i)/ c ;
- our semimeasure on objects does not $*$ -exceed \mathbf{m}^A .

It is enough to construct a computable winning strategy in this game. Indeed, then we can play it against \mathbf{O}' -computable sequence A and the universal semimeasure on strings. Then our moves are computable, and we generate semimeasures on lengths and objects. The winning condition guarantees now that (assuming the limit path A exists) either the opponent's semimeasure along the path is not maximal (so A is not K -trivial), or that \mathbf{m}^A is $*$ -bounded by \mathbf{m} (so A is K -low).

As in the previous theorem, we consider an easier (for us) version of the game where the opponent starts the game by declaring some constant c that he plans to achieve (in the second condition), and we need to beat only this c . If we can win this game with $c = 2^{2^k}$, then we can win the game with $c = 2^k$ using only 2^{-k} total weight, so we can combine these strategies for all k (we also assume that the total weight on objects for k th strategy is also bounded by 2^{-k} , but this is for free, since we need only to $*$ -exceed \mathbf{m}^A without any restrictions on the constant). So it remains to win the game for each c . It is convenient to scale that game and assume that the opponent needs to match our weights on length exactly (not up to $1/c$ -factor), but his total weight is bounded by c (not 1).

Winning the game for $c < 2$. For $c = 1$ the game is trivial, since we require that the opponent's weight along the path is strictly greater than our weight on lengths, so it is enough to assign weight 1 to some length. We start our proof by explaining the strategy for the case $c < 2$.

The idea can be explained as follows. The naïve strategy is to believe all the time that the current path A is final, and just assign the weights to objects according to \mathbf{m}^A , computed based on the current path A . (In fact, at each moment we look at some finite prefix of A and follow the labels that appear on this prefix.) If indeed A never changes, this is a valid strategy: we achieve \mathbf{m}^A , and never exceed the total weight 1 due to the assumption about the labels. But if the path suddenly changes, then all the weight placed because of vertices on old path outside the new path, is lost. If we now follow all the labels on the new path, then our total weight on objects

⁴Formally speaking, we construct two semimeasures on integers; to avoid confusion, it is convenient to call their arguments "lengths" and "objects".

may exceed 1 (it was bounded only along every path, and now we have a part of the old path plus the new path).

There is some partial remedy: we can match the weights only up to some constant (say, use only 1% of what the label asks). This is enough since the game allows us to match the measure with arbitrary constant factor. In this way we can tolerate up to hundred changes in the path (each new path generates new weight at most 0.01), but this does not really help, since the number of changes (of course) is not bounded. Not a surprise, since we did not use the other part of the game, and without this part there is no hope (not all \mathbf{O}' -computable A are low).

How can we discourage the opponent to change the path? We can assign a non-zero weight to some length and wait until the opponent matches this weight along the current path. This is a kind of incentive for the opponent not to leave the vertex where he has put some weight: if the opponent leaves it, he would be forced to waste this weight (and put the same weight along the final path for the second time). After that we could implement a request (label) at this vertex: at least we know that if later the path changes and we lose some weight (by implementing the request not on the limit path), the opponent loses some weight, too. This helps if we are careful enough.

Let us explain the details. It would be convenient to represent the strategy as the set of parallel processes: for each vertex x we have a process P_x that is awake when x is a prefix of the current path (and sleeps when x is not). When awake, the process P_x tries to create the incentive for the opponent not to leave x , by forcing him to increase the weight of some vertex above x . To make the processes more independent (and the analysis simpler), let us assume that for every vertex x some length $l_x \geq |x|$ is chosen, lengths assigned to different vertices are different, and P_x increases only the weight of l_x .

Now we are ready to describe the process P_x . Assume that x carries the request (i, η) that asks to add η to the weight of object i . The process P_x increases the weight of l_x (adding small portions to it and waiting after each portion until the opponent matches this increase along the current path, in some vertex above x). When (and if) the weight of l_x reaches $\varepsilon\eta$ (where ε is some small positive constant; the choice of ε depends on c , see below), the process increases the weight of object i also by $\varepsilon\eta$ and terminates.

The processes P_x for different vertices x run in parallel independently, except for one thing: if the total weight spent by all the processes reaches 1, we terminate all of them (so the increase that would make the total weight greater than 1 is blocked); after that our strategy stops working (in the hope that the opponent would be unable to match already existing weights not crossing the threshold c).

About the small portions: for each vertex x we choose in advance the size δ_x of the portions used by P_x , in such a way that $\sum_x \delta_x < \varepsilon$. (We use here the same small ε as above.) In this way we guarantee that the total loss (last portions that were not matched because the opponent changes the path instead and does not return, so the process is not resumed) is bounded by ε .

It remains to prove that this strategy wins c -game for c close to 2, assuming that ε is small enough. First note two properties that are true by construction:

- the sum of our weights for all lengths does not exceed 1;
- at every moment the sum of (our) weights for all objects does not exceed by the sum of (our) weights for all lengths.

(Indeed, we stop the strategy preventing the violation of the first requirement, and the second is guaranteed for each x -process and therefore for the entire strategy.)

If there is no limit path, the strategy wins the game by definition. So assume that limit path A exists. Now we count separately the weights used by processes P_x for x is on the limit path A , and for x outside A . Since the weights for x are limited by the request in x (with factor ε) and sum of the all requests along A is at most 1, the sum of the weights along A is bounded by ε . Now there are two possibilities: either the strategy was stopped (when trying to cross the threshold) or it runs indefinitely.

In the first case the total weight is close to 1 (at least $1 - \varepsilon$, since the next increase will cross 1, and all the portions δ_x are less than ε). So the weight used by processes outside A is at least $1 - 2\varepsilon$, and if we do not count the last (unmatched) portions, we get at least $1 - 3\varepsilon$ of weight that the opponent needs to match twice: it was matched above x for P_x , and then should be matched again along the limit path (that does not go through x ; recall that we consider the vertices outside the limit path). So the opponent needs to spend at least $2 - 6\varepsilon$, otherwise he loses.

In the second case each process P_x for x on the limit path is awake starting from some moment, and is never stopped, so it reaches its target value $\varepsilon\eta$ and adds $\varepsilon\eta$ to the object i (here (i, ε) is the request in vertex x). So our weights on the objects match \mathbf{m}^A for limit path A up to factor ε , and the opponent loses. (We know also that the total weight on objects does not exceed 1, since it is bounded by the total weight on lengths.)

We therefore have constructed a winning strategy for $2 - 6\varepsilon$ game, and by choosing a small ε can win c -game for every $c < 2$.

Using this strategy on a subtree. Preparing for the recursion, let us look at the described strategy and modify it for using in a subtree rooted at some vertex x . We also scale the game and assume that we have some budget α that we are allowed to use (instead of total weight 1). To guarantee that the strategy does not interfere with other actions outside x -subtree we agree that it uses lengths only from some infinite set L of length and nobody else touches these lengths. Then we can assign $l_y \in L$ for every y in the subtree and use them as before.

Let us describe the strategy in more details. It is composed of processes P_y for all y above x . When x is not on the actual path, all these processes sleep, and the strategy is sleeping. But when path goes through x , some processes P_y (for y on the path) become active and start increasing the weight of length l_y by small portions δ_y (the sum of all δ_y now is bounded by $\alpha\varepsilon$, since we scaled everything by α). A supervisor controls the total weight used by all P_y , and as soon as it reaches α , terminates all P_y . When

the process P_y reaches the weight $\alpha\epsilon\eta$, it increases the weight of object i by $\alpha\epsilon\eta$ (here (i, η) is the request at vertex y). So everything is as before, but with factor α and only on x -subtree.

What does this strategy guarantee?

- The total weight on lengths used by it is at most α .
- The total weight on objects does not exceed the total weight on lengths.
- If the limit path A exists and goes through x , then either
 - the strategy halts and the opponent either fails to match all the weights or spends more than $c\alpha$ on x -subtree; or
 - the strategy does not halt, and the semimeasure on objects generated by this strategy, \ast -exceeds \mathbf{m}^A , if we omit from \mathbf{m}^A all the requests on the path to x .

The argument is the same: if the limit path exists and the strategy does not halt, then all the requests along the limit path (except for finitely many of them below x) are fulfilled with coefficient $\alpha\epsilon$. If the strategy halts, the weight used along the limit path does not exceed $\alpha\epsilon$ (since the sum of requests along each path is bounded by 1), the weight used in the other vertices of x -subtree is at least $\alpha(1-2\epsilon)$, including at least $\alpha(1-3\epsilon)$ matched weight that should be doubled along the limit path, and we achieve the desired goal for $c = 2 - 6\epsilon$.

Remark. In the statement above we have to change \mathbf{m}^A by deleting the requests on the path to x . We can change the construction by moving requests up the tree when processing vertex x to get rid of this problem. One may also note that omitted requests deal only with finitely many objects, so one can average the resulting semimeasure with some semimeasure that is positive everywhere. So we may ignore this problem in the sequel.

How to win the game for $c < 3$. Now we make the crucial step: show how we can increase c by recursively using our strategies. Recall our strategy for $c < 2$, and change it in the following ways:

- Instead of assigning some length l_x for each vertex x , let us assign an infinite (decidable uniformly in x) set L_x of integers; all elements should be greater than $|x|$ and for different x these sets should be disjoint (it is easy to achieve this).
- We agree that process P_x (to be defined) uses only lengths from L_x .
- Again P_x is active when x is on the current path, and sleeps otherwise.
- Previously P_x increased the weight of l_x in small portions, and after each small increase waited until the opponent matched this increase along the current path. Now, instead of that, P_x calls the x -strategy described in the previous section, with small $\alpha = \delta_x$, waits until this strategy terminates forcing the opponent to spend almost $2\delta_x$, then calls another instance of x -strategy, waits until it terminates, and so on. For that, P_x divides L_x into infinite subsets $L_x^1 + L_x^2 + \dots$, using L_x^s for s th call of x -strategy, and using δ_x as the budget for each call.

There are several possibilities for the behavior of x -strategy called recursively. It may happen that it does infinitely many steps. This happens when

x is a part of the limit path, x -strategy never exceeds its budget δ_x , and the entire strategy does not come close to 1 in its total spending. In this case we win the game (the part of the semimeasure on objects due to x -strategy is enough to $*$ -exceed \mathbf{m}^A).⁵

If x is not in the limit path, the execution of x -strategy may be interrupted; in this case we know only that it spent not more than its budget, and that the weight used for objects does not exceed the weight used for lengths. (This is similar to the case when an increase at l_x was not matched because the path changed.)

The x -strategy may also terminate. In this case we know that the opponent used almost twice the budget (δ_x) on the extensions of x , and a new call of x -strategy is made for another set of lengths. (This is similar to the case when the increase at l_x was matched; the advantage is that now the opponent used almost twice our weight!)

Finally, the strategy may be interrupted because the total weight used by x -processes for all x came close to 1. Similar thing happened for the case $c < 2$. After that everything stops, and we just wait until the opponent will be unable to match all the existing weights or forced to use total weight close to 3. Indeed, most of our weight (except for $O(\varepsilon)$) was used not on the limit path and already matched with factor close to 2 there — so matching it again on the limit path makes the total weight close to 3.

Induction step. Now it is clear how one can continue this reasoning and construct a winning strategy for arbitrary c . To get a strategy for some c , we follow the described scheme, and the process P_x makes sequential recursive calls of c' -strategies for smaller c' (the difference should be less than 1, for example, we can use $c' = c - 0.5$). More formally, we recursively define a process $S(c, x, \alpha, L)$ where c is the desired amplification, x is a vertex, α is a positive rational number (the budget), and L is an infinite set of integers greater than $|x|$.⁶ The requirements for $S(c, x, \alpha, L)$:

- It increases only weights of lengths in L .
- The total weight used for lengths does not exceed α .
- At each step the total weight used for objects does not exceed the total weight used for lengths.
- Assuming that the process is not terminated externally (this means that x belongs to the current path, starting from some moment), it may halt or not, and:
 - If the process halts, the opponent uses more than $c\alpha$ on strings that have length in L and are above x .
 - If the process does not halt and the limit path A exists, the semimeasure generated on objects by this process only, $*$ -exceeds \mathbf{m}^A .

⁵This case is called “the golden run” in the original exposition of the proof.

⁶The pedantic reader may complain that the parameter is an infinite sets. It is enough to consider infinite sets from some class, say, decidable sets (as we noted in the previous section), or just arithmetic progressions, they are enough for our purposes. Indeed, an arithmetic progression can be split into countably many arithmetic progressions. For example, $1, 2, 3, 4, \dots$ can be split into $1, 3, 5, 7, \dots$ (odd numbers), $2, 6, 10, 14, \dots$ (odd numbers times 2), $4, 12, 20, 28, \dots$ (odd numbers times 4), etc.

The implementation of $S(c, x, \alpha, L)$ uses recursive calls of $S(c-0.5, y, \beta, L')$; for each y above x a sequence of those calls is made with $\beta = \delta_y$ and L' that are disjoint subsets of L (for different y these L' are also disjoint), similar to what we have described above for the case $c < 3$. \square

18. K -LOW AND ML-LOW ORACLES

There is one more description of K -low (or K -trivial) sequences: they are sequences that (being used as oracles) do not change the notion of Martin-Löf randomness. As almost all notions of the computability theory, the notion of Martin-Löf randomness can be relativized by an oracle a (this means that the algorithm that enumerates covers for an effectively null set, now may use oracle a). In this way we get (in general) more effectively null sets, and therefore less random sequences. However, for some a the class of Martin-Löf random sequences remains unchanged.

Definition 18.1. A sequence a is **MLR-low** if every Martin-Löf random sequence remains Martin-Löf random with oracle a .

The Schnorr–Levin criterion of randomness in terms of prefix complexity shows that if a is K -low, then a is also **MLR-low**. The other implication is also true, but more difficult.

Theorem 18.2. *Every MLR-low sequence is K -low.*

We will prove a more general result.

Definition 18.3. Let a and b be two sequences (considered as oracles). We say that $a \leq_{LK} b$ if

$$K^b(x) \leq K^a(x) + O(1).$$

We say that $a \leq_{LR} b$ if every Martin-Löf random with oracle b sequence is also Martin-Löf random with oracle a .

If an oracle b is stronger in Turing sense (=computes) some oracle a , then b gives larger effectively null sets, smaller set of random sequences, and smaller complexity function, so we have $a \leq_{LR} b$ and $a \leq_{LK} b$. So both orderings are more coarse than the Turing degree ordering.

Using this definitions, we can reformulate the definitions: sequence a is K -low if $a \leq_{LK} \mathbf{0}$ and is **MLR-low** if $a \leq_{LR} \mathbf{0}$. So it is enough to prove the following result:

Theorem 18.4. $a \leq_{LK} b \Leftrightarrow a \leq_{LR} b$ for every a and b .

Proof. Again in one direction (from left to right) it follows directly from the Schnorr–Levin randomness criterion. The other direction is more difficult⁷, and will be split in several steps.

Recall that the set of non-random (in Martin-Löf sense; we do not use other notions of randomness here) sequences can be described using universal Martin-Löf test, as the intersection of effectively open sets

$$U_1 \supset U_2 \supset U_3 \supset \dots$$

⁷This was to be expected. The relation $a \leq_{LK} b$ is quantitative: two functions coincide with $O(1)$ -precision; the relation $a \leq_{LR} b$ is more qualitative, we speak here about yes/no question. One can also consider the quantitative version, with randomness deficiencies, but this is not needed: the relation \leq_{LR} is strong enough.

where U_i has measure at most 2^{-i} . The following observation goes back to Kučera and says that the first layer of this test, the set U_1 , is enough to characterize all non-random sequences.

Lemma 18.5. *Let U be an effectively open set of measure less than 1 that contains all non-random sequences. Then a sequence $a = a_0a_1a_2\dots$ is non-random if and only if all its tails $a_k a_{k+1} a_{k+2} \dots$ belong to U .*

Proof of Lemma 18.5. If a is non-random, then all the tails are non-random and therefore belong to U . In the other direction: represent U as a union of disjoint intervals $[u_i]$ (by $[v]$ we denote the set of all sequences that have prefix v). Then $\rho = \sum 2^{-|u_i|}$ is less than 1. If all tails of a belong to U , then a starts with some u_i , the rest is a tail that starts with some u_j , etc., so a can be split into pieces that are among u_i . The set of sequences of the form “some u_i , then something” has measure ρ , the set of sequence of the form “some u_i , some u_j , then something” has measure ρ^2 , etc. These sets are effectively open and their measures ρ^n effectively converge to 0. So their intersection is an effectively null set and a is non-random. Lemma is proven. \square

The argument gives also the following

Corollary: *a sequence x is non-random if there exists an effectively open set U of measure less than 1 such that all tails of x belong to U .*

This argument can be relativized, so randomness with oracle X can be characterized in terms of X -effectively open sets of measure less than 1: *a sequence x is X -nonrandom if there exists an X -effectively open set U of measure less than 1 such that all tails of x belong to U .* This gives one implication in the following equivalence (we denote here the oracles by capital letter not to mix them with sequences):

Lemma 18.6. *Let A and B be two oracles. Then $A \leq_{LR} B$ if and only if every A -effectively open set of measure less than 1 can be covered by some B -effectively open set of measure less than 1.*

Proof of Lemma 18.6. In one direction (if, \Leftarrow) it follows for the discussion above: if a is not A -random, its tails can be covered by some A -effectively open set of measure less than 1 and therefore by some B -effectively open set of measure less than 1, so a is not B -random.

In the other direction: let U be an A -effectively open set of measure 1 that cannot be covered by any B -effectively open set of measure less than 1. The set U is the union of A -enumerable sequence of disjoint intervals $[u_1], [u_2], [u_3]$, etc. Consider a set V that is B -effectively open, contains all B -non-random sequences and has measure less than 1 (e.g., the first level of universal B -Martin-Löf test). By assumption U is not covered by V , so some interval $[u_i]$ of U is not (entirely) covered by V .

The set V has the following special property: if it does not contain *all* points of some interval, then it cannot contain *almost all* points of this interval: the non-covered part has some positive measure. Indeed, the non-covered part is a B -effectively closed set, and if it has measure zero, it has B -effectively measure zero, so all non-covered sequences are B -non-random, and therefore should be covered by V .

So we found an interval $[u_i]$ in U whose part of positive measure is outside V . Then consider the set $V_1 = V/u_i$, i.e., the set of infinite sequences α such that $u_i\alpha \in V$. This is a B -effectively open set of measure less than 1, so it does not cover U (again by our assumption). So there exists some interval $[u_j]$ not covered by V/u_i . This means that $[u_iu_j]$ is not covered by V . Then we repeat the argument and conclude that non-covered part has positive measure, so V/u_iu_j is a B -effectively open set of measure less than 1, so it does not cover some $[u_k]$, etc. In the limit we get a sequence $u_iu_ju_k\dots$ whose prefixes define intervals not covered fully by V . Since V is open, this sequence does not belong to V , so it is B -random. On the other hand, it is not A -random, as the argument from the proof of Lemma 18.5 shows. \square

Let us summarize where we are now. Assuming that $A \leq_{\text{LR}} B$, we have shown that every A -effectively open set of measure less than 1 can be covered by some B -effectively open set of measure less than 1. And we need to show somehow that $A \leq_{\text{LK}} B$, i.e., $K^B \leq K^A$ up to an additive constant, or $\mathbf{m}^A \leq \mathbf{m}^B$ up to a constant factor. This can be reformulated as follows: *for every lower A -semicomputable converging series $\sum a_n$ of reals there exists a converging lower B -semicomputable series $\sum b_n$ of reals such that $a_n \leq b_n$ for every n .*

So to connect our assumption and our goal, we need to convert somehow a converging lower semicomputable series into an effectively open set of measure less than 1 and vice versa. We may assume without loss of generality that all a_i are strictly less than 1. Then $\sum a_n < \infty$ is equivalent to

$$(1 - a_0)(1 - a_1)(1 - a_2)\dots > 0.$$

This product is a measure of an A -effectively closed set

$$[a_0, 1] \times [a_1, 1] \times [a_2, 1] \times \dots$$

so its complement

$$\{(x_0, x_1, \dots) \mid (x_0 < a_0) \vee (x_1 < a_1) \vee \dots\}$$

is an A -effectively open set of measure less than 1. (Here we split the Cantor space into a countable product of Cantor spaces and identify each of them with $[0, 1]$ equipped with standard uniform measure on the unit interval.) We are finally ready to apply our assumption and find some B -effectively open set V that contains this complement.

Now let us define b_0 as supremum of all z such that

$$[0, z] \times [0, 1] \times [0, 1] \times \dots \subset V$$

This product is compact for every z , and V is B -effectively open, so we can B -enumerate all rational z with this property, and their supremum b_0 is lower B -semicomputable. Note that all $z < a_0$ have this property, so $a_0 \leq b_0$. In a similar way we define all b_i and get a lower B -semicomputable series b_i such that $a_i \leq b_i$. It remains to show that $\sum b_i$ is finite. Indeed, the set

$$\{(x_0, x_1, \dots) \mid (x_0 < b_0) \vee (x_1 < b_1) \vee \dots\}$$

is a part of V , and therefore has measure less than 1; its complement

$$[b_0, 1] \times [b_1, 1] \times [b_2, 1] \times \dots$$

has measure $(1 - b_0)(1 - b_1)(1 - b_2) \dots$, therefore this product is positive and the series $\sum b_i$ converges. This finishes the proof. \square

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